

MATHEMATICAL SURVEY AND APPLICATION OF THE CROSS-AMBIGUITY FUNCTION

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Dedication

This thesis is dedicated to my wife, Jaime.

Abstract

This thesis develops an in-depth mathematical description of the cross-ambiguity function. The cross-ambiguity function is a time (τ) and frequency (ν) analysis technique employed to solve many signal processing problems such as interference mitigation and the location of emitters. The function can be given as:

$$X(\tau, \nu) = \int_{-\infty}^{\infty} s_1(t) s_2^*(t + \tau) e^{j2\pi\nu t} dt$$

where the convolution of two transmissions is in analytic signal format. $X(\tau, \nu)$ is the peak energy or power, where τ is the time delay and ν is the frequency off-set (Doppler). $S_1(t)$ is the signal received at one collector, and $s_2^*(t + \tau)$ is the signal (complex conjugate) received at the second collector. Finally, $e^{j2\pi\nu t}$ is the Fourier artifact containing the frequency off-set. Thus, the realm of the cross ambiguity function lies predominantly in the field of communications and electrical engineering where systems design is of importance. As such, the mathematical treatment of the cross-ambiguity function is brief, and is often presented with little detail in order to primarily fulfill engineering goals in the literature. This leaves the reader with subtle, but important gaps in understanding, such as, how convolution takes place, differences in the complex envelope and analytic signals, the Fourier series, and the use of complex conjugates. This thesis provides the mathematical foundation and concepts to more completely illustrate the cross-ambiguity function's characteristics. There are many signal processing problems that can be used to demonstrate the cross-ambiguity function such as the matched filter, system design, noise reduction, and geolocation. This thesis selects collection of an emitter since the inherent geometry of the problem provides the clearest illustration of the function's time and frequency operations. Upon it mathematical concepts such as convolution, correlation, the work of Euler, the complex conjugate, Hilbert transform, the Fourier transform, and advanced integration techniques are presented. Further, the cross-ambiguity function is applied to the case of a square pulse emitted from a signal slow moving emitter and collected from to disparate collectors assumed to be moving at different speeds. This framework sets the stage not only for clarity of the geolocation problem, but a more clear understanding of time and frequency analysis. Finally, important aspects of the cross-ambiguity function are demonstrated in MATLAB.

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1. Introduction

A clear, in-depth, and comprehensive mathematical treatment of the cross-ambiguity function will be presented. This thesis will clearly discuss and document the cross-ambiguity function's underlying mathematical concepts such as convolution in time and frequency, more specifically, what we will call time delay and Doppler shift (frequency off-set). An important concept, the analytic signal, will also be presented and as such lead to discussion of Fourier and Hilbert transforms as well as Euler's definition of complex signal components. Reaching a peak between the time delay and Doppler shift demonstrates the most likely time and frequency features of the geolocation model. The thesis presents the basic geolocation model of a relatively stationary emitter and two moving collectors typical of search and rescue operations. The emitter construction will be a rectangular pulse.

Several assumptions are made. The first is that additive white Gaussian noise (AWGN) will be trivial to the solution and this will be explained more fully in the body of the thesis. In brief, AWGN is constant and low-level broadband background noise chiefly from natural sources. Our signals will be clearly discernible above this noise floor. Further, it will be assumed that the pulse collected by the collectors will be similar. Detection as such will be assumed, the problem of the thesis is to demonstrate the cross-ambiguity function insofar as geolocation is concerned. Therefore, the time and frequency features will be correlated to produce the highest energy peak and thus the emitter's location. That is, it will also be assumed that we know the location of the collectors – only the location of the emitter is unknown. Also, we will assume that the frequency off-set is constant across the width of the pulse. The importance of this effort lies in the comprehensive and detailed survey of the mathematical, not engineering, aspects of the cross-ambiguity function. Results are often presented *deus ex machina* with no fundamental treatment of the concepts underlying their demonstration. For example, the complex conjugate is often cited a part of the convolution process, but little or no attempt is made to explain its appearance.

2. Literature Review

The goal of the literature review was to support this thesis' effort to enhance the understanding of the cross-ambiguity function by integrating a wide range of mathematical concepts into an engineering framework. A comprehensive review of the literature was conducted using the library and Internet. It was found that the literature fell into two basic categories: mathematical literature that concentrated on the concepts underlying the function, and engineering literature that focused on a specific aspect of it. As described by Dominguez [11] in his history of convolution, the familiar integral

equation form of convolution was first discussed by Volterra [41]. This integral equation took the form

$$\int_0^t f(t - \tau)g(\tau)d\tau = \dot{f} \dot{g}(t)$$

where $\dot{f} \dot{g}(t)$ is “la composition de deux fonctions f et g ” - the composition of two functions f and g . The notation was later formalized by Doestch [10] as

$$L^{-1}(f_1 f_2) = \int_0^t F_1(\tau)F_2(t - \tau)d\tau$$

with the right hand side of the equation denoted as $F_1 * F_2$. Convolution is the amount of overlap between two distinct functions [45], and has many practical applications in imaging, acoustics, digital signal processing and radar. Convolution, by itself, is given satisfactory treatment in the mathematical literature by Bracewell [5], Debnath and Mikusinski [8], and Rahman [29].

Relating to this work, convolution was applied in the area of the complex signal by Ville in his ground-breaking document [40]. Ville compared the characteristics between two signals using convolution as his basic operator to determine the level of shared characteristics, especially in time and frequency. His mathematical treatment is impressive. However, the overwhelming goal of the document is an engineering discussion of the complex signal in time-frequency analysis. Later, Woodward, in his seminal work [47], applied it to the processing of two radar reflections. Here, with Ville in mind, Woodward presented a function that analyzes the time and frequency components of the radar signal. The purpose was to determine the amount of shared characteristics by investigating the peak energy when time and frequency are considered in two dimensions. Called cross-correlation, this is what has become known as the cross-ambiguity function. In [47] he discusses convolution, but it is a cursory discussion leading up to a more applied discussion of the cross-ambiguity function from strictly an engineering standpoint.

Among other areas of inquiry by Moura and Rendas [25], with the advent of automated computer processing, much literature has been devoted to making the cross-ambiguity function's processing more efficient, Stein [36] and Yatrakis [49]. Stein, uses the basic Woodward function in minimizing the processing burden. All documents focus on their application, and generalize the mathematical foundation upon which they were developed.

3. Convolution

3.1 Preamble

In general mathematical terms, convolution is the combination of two functions resulting in a third function [35]. While convolution has many different fields of application, it has a common use in that it determines the amount of overlap between two functions. It can be applied to many areas such as engineering, image and signal processing, acoustics, optics [42], and biology to compare seemingly similar features.

3.2 History

Convolution in its modern form has been mainly attributed to Italian mathematician Vito Volterra. Other 17th- to 19th-century mathematicians employed similar concepts in their particular fields such as D'Alembert in Taylor's series, Fourier and Dirichlet in the Fourier series, Euler in differential equations, Abel and Louville in integral equations, and Reimann in fractional calculus [11]. Some even credit Doetsch, who referred to the process of convolution, in a German word used by Hilbert [24] – as Faltung (folding), with its current use. (This concept of a Faltung Integral [29], "folding", "flipping", or "reversing" a function will be addressed in later sections.) Each researcher was in some way trying to describe the behavior of a system over time through their particular application. As discussed earlier, it was Volterra's work in integral equations that presented and codified the familiar form of the convolution integral used today. In his book [41] Volterra takes the integral of the product of two functions in what he calls the composition of the first kind to produce a resultante [11]. Represented here again, the composition is

$$\int_0^t f(t - \tau)g(\tau)d\tau = f \dot{g}(t).$$

This has become the standard integral equation form of the real convolution integral where two functions are composed over a specific time period, t . The variable, τ , is the integration variable. As the integral suggests, we will be examining each time delay value from zero to t for a functional combination exhibiting the most commonality. A central and common feature to all the work of mathematicians in this area of composition products is the aspect of the time-shifted function. In order to examine the historical behavior of a system, a plus (+) or (in this case) minus sign (-) operator must be introduced. Then the product of two functions is found and finally summed or integrated over a specific time period. Thus, we highlight the idea of a time shift and off-set that becomes one of the main analytical features of the cross-ambiguity function.

3.3 Basic Principles

Convolution is the mathematical foundation of the cross-ambiguity function. For our purposes, convolution is an operation that composes [5] an output by combining an input that has been weighted and summed by another function over a defined range of values. Thus, we are viewing it in a mathematically traditional [35] way where a result (the output) is the function of one or more variables. One input function and another weighting function are, at each value, multiplied and the value(s) are summed to produce a result that represents a standard of overlap. The output function represents these results at each value(s) from which one can determine the character of the input. The area of digital signal processing, which this thesis invokes, is especially amenable to convolution since signals can be digitized (sampled) into discrete values. Signals from different sources can be compared through the convolution method because their individual samples can be easily manipulated by multiplication and integration. Not only can the output be viewed numerically, but the convolution method can be visualized quite naturally.

4. Discrete-Time Convolution

At this point, it is natural to begin discussing the mathematical mechanics of convolution using the discrete case. The discrete case has intrinsic illustrative value since, as the aforementioned paragraph states, in digital signal processing, signal samples are the basic elements processed in convolution. Thus, discussing the discrete case has not only conceptual mathematical value, but applied value as well.

In discrete-time convolution, the convolution takes the product of two functions and then sums them over a defined time period. The discrete-time convolution formula is

$$(x * h)[n] = y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

where $y[n]$ is the output function, $x[k]$ is the input function, and $h[n - k]$ is the weighting function [1], [27] and [32]. The value n is an important variable. The variable n is an upper limit on the indices and often represents the specific time [32] period in question. In our application, n is a specific time at which samples from the two signals $x[k]$ and $h[n - k]$ are multiplied. The variable index k performs indexing for each k th [37] value up to and including, n , i.e., ($k \leq n$), over which products of the two functions are summed. As such, k performs the accumulation for the convolution. Thus, the discrete convolution formula performs calculations for each specific instance of time, n . Each sample at the specific time n of the output signal is composed of the summation of

products indexed at each, k . For example, if the output signal is composed of nine discrete time samples, each discrete time sample is its own particular instance of time, n , so there are nine different n 's. For instance, the following example using the discrete time formula illustrates this concept.

$$y[0] = \sum_{k=0}^0 x[k]h[0-k] \quad \text{discrete time sample 1}$$

$$y[1] = \sum_{k=0}^1 x[k]h[1-k] \quad \text{discrete time sample 2}$$

*

*

*

$$y[8] = \sum_{k=0}^8 x[k]h[8-k] \quad \text{discrete time sample 9}$$

Thus, the output signal is composed of $y[0], y[1], y[2], y[3], y[4], y[5], y[6], y[7]$ and $y[8]$ - the formulas above. So we should not, for example, confuse $y[8]$ as representing an entire sequence of samples, although some authors do use such notation. This clearly is a matter of notation, but one that is frequently employed. Other notations express not just the transformation of the function(s) in terms of a specific time, n , but may use "." to denote an entire sequence of $h[n-k]$ and $y[n]$, for example, $y[\cdot] = T\{x[\cdot]\}$ [14]. So, we could write that $y[\cdot] = y[0], y[1], y[2], y[3], y[4], y[5], y[6], y[7]$ and $y[8]$.

The cross-ambiguity function represents a continuum of values each related by a time-shift demonstrating a convergence of product and sums. As we will see, the convergence of values represented by this sequence of values forms an ideal characteristic maximum value at some specific discrete time sample. All other discrete time samples should be less than this maximum value, or as we will call it a characteristic "peak".

4.1 Upper and Lower Indices

Mathematically, the lower index of k , the start of the indexing, is the first instance of time where the first non-zero value of convolution first occurs, i.e., the first non-zero instance where samples from the two different functions (signals) are multiplied and summed. The upper index n is the last instance of a non-zero product of samples

between the two functions [32]. Thus, the products of the output signal are zero below the lower limit and above the upper limit [32].

The upper and lower indices are the samples in time where not only do we find a series of non-zero products and sums, however, these also define the length of the output [32] function, $y[n]$. The output signal is comprised of all these non-zero products. We can then determine the length of the output signal [35] by finding the length of the input signal and weighting function. This relation [4] and [32] can be given as

$$M + N - 1 = L$$

where M is the length of the input signal $x[k]$, N is the length of the weighting action signal $h[n - k]$, and L is the length of the output signal $y[n]$. The length refers to the number of samples in each signal. Thus, for example, if the length of $x[k]$ is 5, and the length of $h[n - k]$ is 6, then the length of $y[n]$ is $5 + 6 - 1 = 10$. Given the length of the output signal we can determine given either the first or last instance of non-zero products the starting or ending time of convolution. For example, examining the output signal's $y[0]$ sample, we can determine the starting time of convolution since one of the benefits of digital signal processing is the time-tagging of data elements not only of a pulse of energy, but even the behavior of that energy within the pulse. Suppose we call the starting time t_0 . Knowing the length, L , of the output signal, we can then determine the time at the end of L by the notation t_L for sample $y[8]$, for example.

4.2 Application

As an illustration of our discrete-time convolution, P.M. Woodward lays the groundwork for the cross-ambiguity function himself in this way in his seminal work on the cross-ambiguity function [47]. In the reference, Woodward introduces the convolution sum for two functions each with possibly a different probability distribution. The following illustrates our discussion.

$$R(u) = \sum_r P(r)Q(u - r)$$

where $P(r)$ and $Q(s)$, with $s = u - r$, are two independent random quantities. The variable u is a fixed value, and the variables r and s are not fixed. As such, the result, R , is influenced by an input, $P(r)$ which was weighted by the shifted quantity $Q(u - r)$ and then summed over a range of values, r . In his work, Woodward explains this in context of the statistical distributions of each function. As such, we are interested in the finding the statistical distribution of the output $R(u)$ using the product rule [47]. The following example is based on Woodward's statistical description of convolution [47]. He examines a set of eight pencils with the attributes of color and hardness.

A	A	A	B	B	B	B	C
J	J	K	J	J	J	K	K

Where A is red, B is black, C is blue, J is hard, and K is soft. If we were to choose a pencil based on color only and randomly chose A, the unconditional probability that we chose A was $\frac{3}{8}$. Having selected A, the probability that the pencil is hard is now $\frac{2}{3}$ and the probability that it could be soft is $\frac{1}{3}$. Thus, these are the new conditional probabilities of hardness and softness both based upon having chosen a red pencil. Assume, we have chosen a hard pencil, J. We can now state, as does Woodward in familiar probability notation

$$P(A,J) = P(A)P_A(J) = \frac{3}{8} \cdot \frac{2}{3} = \frac{1}{4}$$

Thus, Woodward is describing the probability distribution of a pencil having these attributes [47]. Regardless of the application, the convolution framework is the same. Convolution determines how much information the output receives from the input after the input has been weighted.

4.3 Summary

In a broad, conceptual sense we will be investigating the value of the product and summation at each specific n to determine the point of most commonality across the length of the output function. In our cross-ambiguity function analysis, we will be testing the time delays, k , across a period of time. The k yielding the highest degree of commonality is usually the time delay of interest. Moreover, the time period in our context will be just not confined by where non-zero products begin and end. Based on our cross-ambiguity function's operating framework it is expected to include times at which convolution products are zero. In essence, the ambiguity function is searching for an optimum time delay value across a specific time range.

5. The Linear Time-Invariant System

Convolution is closely tied to the concept of the linear time-invariant (LTI) system. It is a key intersection between the discussion of convolution and our signal processing context. A linear time-invariant system is an ideal, input-output transformation model that has two characteristics.

1. Linearity. The system must provide for linearity between the input and output, such that their properties are:

a. Scaling. Where multiplying the input by a constant scalar yields the same corresponding scaling of the output [16] and [33]

$$T\{ax[n]\} = aT\{x[n]\} = ay[n]$$

b. Additive Superposition. Where summing of the input yields the same corresponding summing of the output

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

2. Time-invariance. Where a time shift [16] and [33] in the input yields a corresponding shift in the output

$$T\{x[n - k]\} = y[n - k]$$

The linear time-invariant system is fully described by a convolution sum. In convolution, the idea of shifting a function over some time period to gain a historical perspective on the behavior of a system provides the best mathematical model to study linear time-invariant systems since these systems preserve time shifts in their operations. Further, we now introduce the idea of impulses and the impulse response to see direct applicability between a linear time-invariant system and convolution.

5.1 Impulses and Impulse Response

Discrete-time signals are composed of, what are called, *impulses* (excitations) [1]. An impulse is a discrete spike of energy representing an ideal discrete signal sample and forms the basic elements for signal processing analysis. The notation, $\delta[u]$, is used to describe this single pulse of energy. And as such introduces Dirac's delta "improper" function defined [5] as

$$\delta[n] = \begin{cases} +\infty, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(n) dn = 1.$$

The Dirac delta function is not formally a function since at one point we have an undefined value, but its integral is one over a time period. It is used in digital signal

processing to conveniently represent a single discrete sample of a signal in time. As such, the Dirac function is conveniently used to model a single impulse of energy. We will assume the Dirac "function" has a unit area of one by the definition above. As such, we will call the input, $\delta[n]$, the *unit impulse function* and it is depicted in Figure 5.1 below.

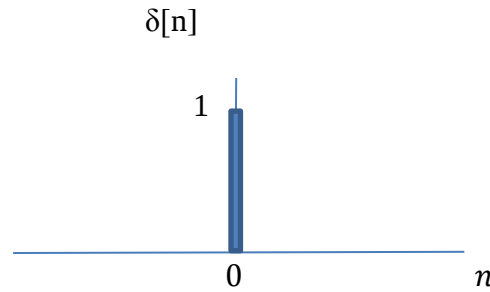


Figure 5.1

For clarification and completeness, some authors use the Kronecker Delta function [27] and [33] which is closely related to the Dirac function, and differs primarily in that the value of the function is "1" at $n = 0$ rather than " ∞ ".

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The Kronecker delta function can be used to represent the unit impulse function in the same way as Figure 5.1.

We now introduce the idea of an *impulse response* which together with the unit impulse function forms the output of a system's behavior. Ultimately, knowing the output function and the impulse response of a system, one can readily determine the characteristics of the input function.

An input-output diagram of a linear time-invariant system [35] is shown in Figure 5.2. The output [14], the *impulse response* which is usually denoted by $h[n]$ [27], is the result of an input, the unit impulse $\delta[n]$, transformed by the linear time-invariant system by a scalar multiple, linear additive constant or time shifted value [1] and [32].

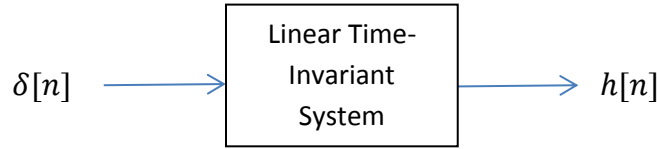


Figure 5.2

Graphically, this relationship may be visualized in Figure 5.3. For illustration, the unit impulse has been multiplied (transformed by the LTI) by a scalar factor of 0.75. The LTI is the transformation function, T .

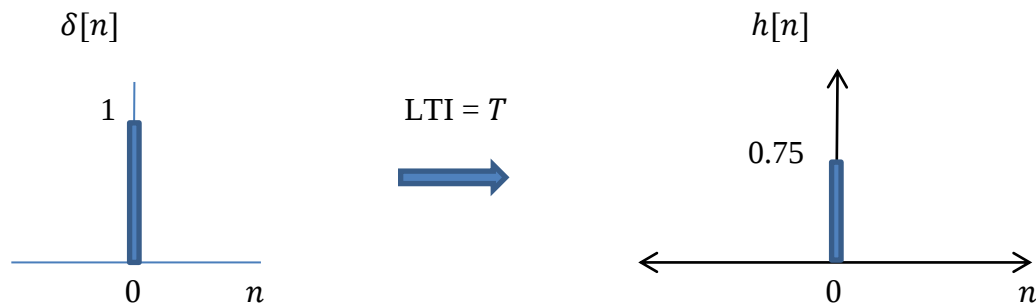


Figure 5.3

and mathematically as

$$T\{\delta[n]\} = h[n]$$

The impulse response, $h[n]$, in Figure 5.3 is a simple arbitrary representative transformation of the unit impulse by the linear time-invariant system. The impulse response of the system is the output that results in response to a unit impulse input [32]. As stated above, knowing the impulse response $h[n]$, you can fully characterize the entire linear time-invariant system. The linear time invariant system can now be used to construct a discrete-time sequence of signal samples. Taking $x[n]$ as my output function, one can construct it [1] and [27] by the following manner

$$x[n] = \sum_{k=-\infty}^{\infty} h[k]\delta[n-k]$$

This is exactly the same form of the discrete-time convolution formula introduced earlier representing the product and then sum of an individual signal sample, n . To demonstrate [32] how a sequence of discrete time samples in the output is represented by a sequence of weighted and shifted unit impulses the following is illustrated in Figure 5.4.

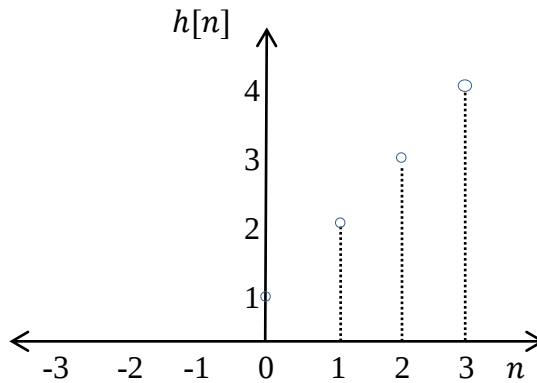


Figure 5.4

Figure 5.4 represents the output of a discrete time signal comprised of specific samples of n at 0, 1, 2 and 3 each with their own arbitrarily chosen magnitudes. Each of these samples of n can be represented by the product of a unit impulse by a weighted function. In the following demonstration we compose an $x[n]$ sequence as seen in Figure 5.4 using unit impulses. Figures 5.5 to 5.8 represent the incremental building of the output signal as the delta function is shifted and scaled.

for $n = 0$

$$\begin{aligned}
 x[0] &= \sum_{k=0}^0 h[k] \delta[0 - k] = h[0] \delta[0 - 0] \\
 &= 1 * \delta[0] \\
 &= 1 * 1 \\
 &= 1
 \end{aligned}$$

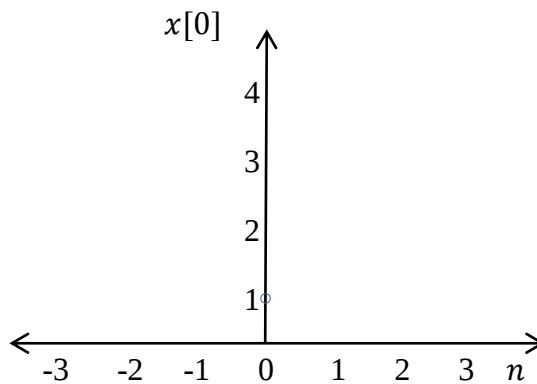


Figure 5.5

for $n = 1$

$$x[1] = \sum_{k=1}^1 h[k] \delta[1-k] = h[1] \delta[1-1]$$

$$= 2 * \delta[0]$$

$$= 2 * 1$$

$$= 2$$

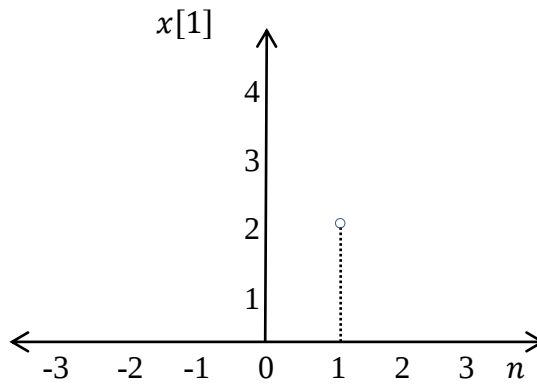


Figure 5.6

for $n = 2$

$$x[2] = \sum_{k=2}^2 h[k] \delta[2-k] = h[2] \delta[2-2]$$

$$= 3 * \delta[0]$$

$$= 3 * 1$$

$$= 3$$

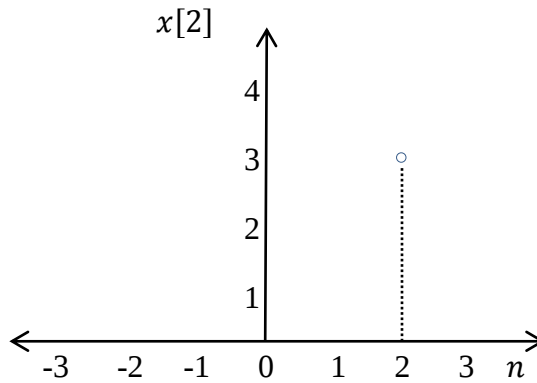


Figure 5.7

for $n = 3$

$$\begin{aligned}
 x[3] &= \sum_{k=3}^3 h[k] \delta[3 - 0] = h[3] \delta[3 - 3] \\
 &= 4 * \delta[0] \\
 &= 4 * 1 \\
 &= 4
 \end{aligned}$$

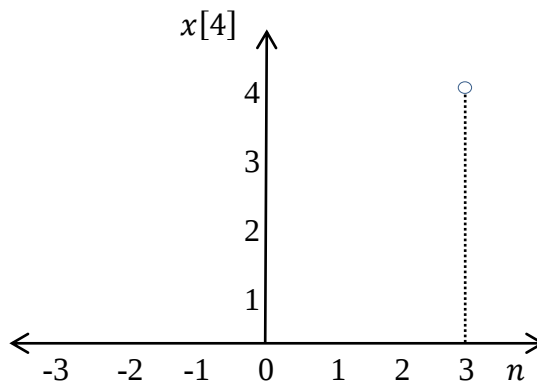


Figure 5.8

We can see that $x[n]$ provides the scaling and δ provides the unit impulse and time shift for each, n . Superimposing these four graphs over each other reconstitutes $x[n]$ in Figure 5.4. Again, notice that in each specific sample n for Figures 5.5 to 5.8, there has been a corresponding shift of Dirac's delta function to the right for each signal sample, n .

Thus, we have featured one of the most important elements of the convolution process, the time shift. Here we are investigating the history of the signal sample from $n = 0$ to $n = 3$ in constituting an entire sequence of discrete signal samples. Mathematically, the convolved output signal, $x[n]$, can be written [32], for all n in this case, as

$$\begin{aligned}(h * \delta)[n] &= x[n] = h[0]\delta[n] + h[1]\delta[n - 1] + h[2]\delta[n - 2] + h[3]\delta[n - 3] \\ &= h[0] + h[1] + h[2] + h[3].\end{aligned}$$

Thus, convolution can be used as a tool to construct a signal from a series of weighted and time-shifted values. In his book, Bracewell best describes this action as a "superposition of characteristic contributions" [5] where infinitesimal contributions from approximating rectangles of a function are added in overlapping fashion revealing a final superimposed product representing the convolution of the two functions.

6. The Basic Convolution System

In this next section, we use $x[n]$ as an example of an input signal into a LTI system. Figure 6.1 shows this general transformation system [35]. The input signal $x[n]$ is characterized by the impulse response, $h[n]$, yielding the output signal $y[n]$.

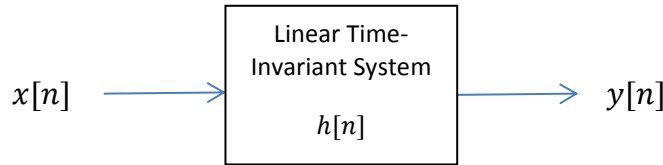


Figure 6.1

This relationship can be expressed as

$$T\{x[n]\} = y[n]$$

where the transformation function T is created by $h[n]$, the impulse response. Here a more complete picture of discrete-time convolution is presented. We are now expressing our output signal, $y[n]$, in terms of two functions $x[n]$ and $h[n]$. As presented earlier, the full discrete-time convolution formula is

$$(x * h)[n] = y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

The input signal $x[n]$ is now represented by $x[k]$. The impulse response of the linear time-invariant system, $h[n]$ is now represented by $h[n - k]$ with a time-shift of n since the system is defined as time-invariant. The $y[n]$ represents one signal sample composed of the products and sums of k th samples from $x[k]$ and $h[k]$. Reference [32] provides an excellent algorithmic description of the discrete-time convolution process and is presented here with modification. The purpose of this step-by-step analysis is to demonstrate the product and sum actions of convolution and enable the reader gain better understanding through the exercise provided in the following section.

Discrete-Time Convolution Steps

1. Time shift $h[k]$ by n to form $h[n - k]$ (causes folding)
2. Multiply $x[k]$ and $h[n - k]$ for all values of k
3. Sum all products $x[k]h[n - k]$ for all k to find $y[n]$

The idea of time reversal and folding will become clearer as we construct the output signal, $y[n]$, sample-by-sample from the input signal and impulse response of the system. It can be discerned through the process of constructing each n -by- n sample of the output signal by the $n - k$ subtraction and indexing by k up to n causing the $h[k]$ values to be multiplied and summed in reverse order. This is what is meant by examining the historical behavior of the system.

6.1 Software Tools Demonstration

The aforementioned algorithm can be demonstrated neatly using software tools. First, the algorithm will be instantiated in Excel, and then illustrated using MATLAB. Excel is a powerful spreadsheet program used in solving engineering and mathematical problems. I will demonstrate discrete convolution two ways in Excel.

The first employed is a straightforward sum and product statement combined with careful placement of $x[n]$ using the set of magnitudes representing the amplitude of each sample $\{1, 2, 3, 4\}$ and $h[n]$ using the set of magnitudes $\{2, 1, 4, 3\}$ values within the spreadsheet. The benefit of this approach is its simplicity. Looking at the input data and the resulting $y[n]$ values lends itself to a clear, but still general understanding of the process. The Excel SUMPRODUCT command is used, and as this case is relatively simple the specific command, input data, and output data are presented here. The Excel command used is SUMPRODUCT(A\$2:A\$13,B2:B13) where the A – values represent one signal (or the input signal $x[n]$) and the B – values represent the other signal (or the impulse response $h[n - k]$). As indicated the $x[n]$ values are fixed and the $h[n - k]$

values are variable, see Figure 6.2 below. The $y[n]$ values were produced with the SUMPRODUCT command. As you can see from the data, the signals are given some arbitrary spacing between each other, and they include zeroes to indicate they are not only a finite sequence – see Section 4.1, but to also provide nulls in the SUMPRODUCT to produce the output as necessary. Notice that $x[n]$, is the fixed signal, and that $h[n]$ has been flipped and shifted to become $h[n - k]$. We can visualize the algorithm convolving the signals as we inspect the two Excel columns seeing the two finite data sets approaching each other as depicted in Figure 6.2 below.

$x(n)$	$h(n)$	$y(n)$
0	0	0
4	0	0
3	0	0
2	0	2
1	0	5
0	0	12
0	0	22
0	2	22
0	1	25
0	4	12
0	3	0
0	0	0

Figure 6.2

The $y[n]$ values in Figure 6.2 are the set of magnitudes representing the amplitude of each sample $\{2, 5, 12, 22, 22, 25, 12\}$.

The second application of an Excel spreadsheet solution is provided by reference [12]. The same input signal vectors will be used as in the previous example. The Convolution Table of Two Sequences, Figure 6.3, follows the same form as [12]. The table clearly demonstrates the shifting of the impulse response signal $h[n]$ and the creation of $y[n]$. The $x[k]h[n - k]$ computations are the product of those two individual factors referenced from the $h[n - k]$ row and $x[k]$ column figures where numerical flipping of the impulse response signal is affected. The summation of the individual output signal samples are recorded in the $y[n]$ row.

Convolution Table of Two Sequences									
First Sequence: $x(n)=[1,2,3,4]$									
Second Sequence: $h(n)=[2,1,4,3]$									
n	0	1	2	3	4	5	6	k	x(k)
$h(n)$	2	1	4	3				0	1
$h(n-1)$		2	1	4	3			1	2
$h(n-2)$			2	1	4	3		2	3
$h(n-3)$				2	1	4	3	3	4
$x(0)h(n)$	2	1	4	3	0	0	0		
$x(1)h(n-1)$	0	4	2	8	6	0	0		
$x(2)h(n-2)$	0	0	6	3	12	9	0		
$x(3)h(n-3)$	0	0	0	8	4	16	12		
$y(n)$	2	5	12	22	22	25	12		

Figure 6.3

The product rows of $x[k]h[n - k]$ demonstrate with unique clarity how the sum of each product occurs for each of the output signal's sample in the respective vertical column. The products for each sample are added vertically. The increasing progression, peaking, and then decrease in the magnitude of the output signal's amplitudes is especially instructive.

The second way in which the discrete-time convolution algorithm will be demonstrated is using MATLAB. MATLAB is used extensively in this thesis, and also provides a clear description of the discrete-time convolution process within its own programming syntax. Further, MATLAB will provide the computations for continuous convolution in later sections and this demonstration provides a convenient introduction. Also, MATLAB is a fourth-generation programming language for numerical computation and Figure 6.4 instantiates the commands of the discrete-time convolution process in a MATLAB script [9] with remarks - %. The same signal vectors defined in the two Excel exercises are used here as well.

```

x = [1 2 3 4]; % input signal x[n]
t1 = length(x); % length of x[n]
h = [2 1 4 3]; % impulse response h[n]
t2 = length(h); % length of h[n]
y = conv(x,h); % convolution of the two signals
t = 1 : 1: t1 + t2 - 1; % length of y[n]
y % display sequence of output signal sample magnitudes
y =
2     5    12    22     2    25    12

```

Figure 6.4

The MATLAB code, like the Excel demonstrations, illustrates not only the simplicity of defining our signals as simple vectors, however, the length of each signal is easily instantiated and used to compute the length of the output signal $y[n]$. As we introduced in Section 4.1, the length of the output signal is $M + N - 1$. In this current MATLAB case [9] $t1 = M$, $t2 = N$, and we can see the length is seven samples. Even in MATLAB, the process of computing the output signal is clear and pre-packaged as its own command “conv”. As we discuss continuous convolution, this command will also be used with some important additions.

7. Discrete-Time Convolution Exercise

The purpose of the following exercise is to illustrate the basic concepts involved in convolution. Like our introduction, it uses the discrete-time case as a demonstration. We will begin by examining each sample of the output signal and determine how it was created using our algorithm as a basis. By doing so, a clear pattern will appear that supports the idea of non-products, folding, time delay, and signal length. We use as our input signal, $x[k]$, the previous output signal, $x[n]$ as demonstrated in the impulse function exercise. Our pre-time-shifted impulse response signal, $h[n]$, is represented by Figure 7.1. Notice the values of the (arbitrarily chosen) magnitudes at each sample of $h[n]$ takes on. These will be multiplied into successive values of $x[k]$ as time shifts for $h[n]$.

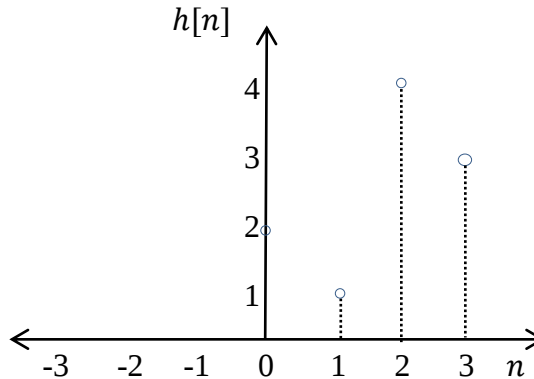


Figure 7.1

We begin by calculating the sample at the specific time (lower limit) $n = 0$ using the discrete time convolution formula and continue to an upper limit of $n = 6$ [27], [32] and [33].

for $n = 0$ (the first instance of non-zero products)

$$\begin{aligned}y[0] &= \sum_{k=0}^0 x[k]h[0-k] = x[0]h[0-0] \\&= x[0]h[0] \quad (\text{composed of one non-zero product}) \\&= 1 * 2 \\&= 2 \quad (\text{the amount of overlap at this sample})\end{aligned}$$

for $n = 1$ (the next non-zero product...)

$$\begin{aligned}y[1] &= \sum_{k=0}^1 x[k]h[1-k] = x[0]h[1-0] + x[1]h[1-1] \\&= x[0]h[1] + x[1]h[0] \quad (\text{two non-zero products...}) \\&= 1 * 1 + 2 * 2 \\&= 5 \quad (\text{the amount of overlap at this sample...})\end{aligned}$$

for $n = 2$

$$\begin{aligned}y[2] &= \sum_{k=0}^2 x[k]h[2-k] = x[0]h[2-0] + x[1]h[2-1] + x[2]h[2-2] \\&= x[0]h[2] + x[1]h[1] + x[2]h[0] \\&= 1 * 4 + 2 * 1 + 3 * 2 \\&= 12\end{aligned}$$

for $n = 3$

$$\begin{aligned}y[3] &= \sum_{k=0}^3 x[k]h[3-k] = x[0]h[3-0] + x[1]h[3-1] + x[2]h[3-2] \\&\quad + x[3]h[3-3] \\&= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] \\&= 1 * 3 + 2 * 4 + 3 * 1 + 4 * 2 \\&= 22\end{aligned}$$

for $n = 4$

$$\begin{aligned}y[4] &= \sum_{k=0}^4 x[k]h[4-k] = x[0]h[4-0] + x[1]h[4-1] + x[2]h[4-2] \\&\quad + x[3]h[4-3] + x[4]h[4-4] \\&= x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0] \\&= 1 * 0 + 2 * 3 + 3 * 4 + 4 * 1 + 0 * 2 \\&= 22\end{aligned}$$

for $n = 5$

$$\begin{aligned}y[5] &= \sum_{k=0}^5 x[k]h[5-k] = x[0]h[5-0] + x[1]h[5-1] + x[2]h[5-2] \\&\quad + x[3]h[5-3] + x[4]h[5-4] + x[5]h[5-5] \\&= x[0]h[5] + x[1]h[4] + x[2]h[3] + x[3]h[2] + x[4]h[1] \\&\quad + x[5]h[0] \\&= 1 * 0 + 2 * 0 + 3 * 3 + 4 * 4 + 0 * 1 + 0 * 2 \\&= 25\end{aligned}$$

for $n = 6$ (the last non-zero sample)

$$\begin{aligned}y[6] &= \sum_{k=0}^6 x[k]h[6-k] = x[0]h[6-0] + x[1]h[6-1] + x[2]h[6-2] \\&\quad + x[3]h[6-3] + x[4]h[6-4] + x[5]h[6-5] \\&\quad + x[6]h[6-6] \\&= x[0]h[6] + x[1]h[5] + x[2]h[4] + x[3]h[3] + x[4]h[2] \\&\quad + x[5]h[1] + x[6]h[0] \\&= 1 * 0 + 2 * 0 + 3 * 0 + 4 * 3 + 0 * 4 + 0 * 1 + 0 * 2 \\&= 12\end{aligned}$$

Thus, the formula reveals that convolution is a progression of sequential multiplication and summation [1]. Looking more closely at the figures, we can see a pattern of shifting and consequential overlap occurring that reflects the weighting of the input by the impulse response signal [32]. For those elements outside the construct of the input signal or impulse response a padding factor of zero is given since multiplying by nothing is equivalent to multiplying by zero. By eliminating the highlighted zero products from the calculations, let us simplify the above for visualization purposes.

$$y[0] = x[0]h[0]$$

$$y[1] = x[0]h[1] + x[1]h[0]$$

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$y[4] = x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0]$$

$$y[5] = x[0]h[5] + x[1]h[4] + x[2]h[3] + x[3]h[2] + x[4]h[1] + x[5]h[0]$$

$$y[6] = x[0]h[6] + x[1]h[5] + x[2]h[4] + x[3]h[3] + x[4]h[2] + x[5]h[1] + x[6]h[0]$$

These equations simplify as the following

$$y[0] = x[0]h[0]$$

$$y[1] = x[0]h[1] + x[1]h[0]$$

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$y[4] = \quad \quad \quad x[1]h[3] + x[2]h[2] + x[3]h[1]$$

$$y[5] = \quad \quad \quad \quad \quad \quad x[2]h[3] + x[3]h[2]$$

$$y[6] = \quad \quad \quad \quad \quad \quad \quad \quad \quad x[3]h[3]$$

The purpose of this is to demonstrate folding or time reversal. The shifting of the impulse response values by $h[n-k]$ causes a reversal effect of the those values as can be discerned by examining the make-up of each output $y[n]$. For example, for $y[1]$ the first product includes $h[1]$ then the next product is $h[0]$. For $y[2]$, the first product has the factor $h[2]$, the second factor $h[1]$ and the third factor $h[0]$. This process of reversal,

sometimes called “folding” lends itself to visualizing convolution in a more intuitive way. Figure 7.2 illustrates the folding [2] of the n values across the $h[n]$ axis, where $h[n] = h[-n]$ is a symmetric and even function [37].

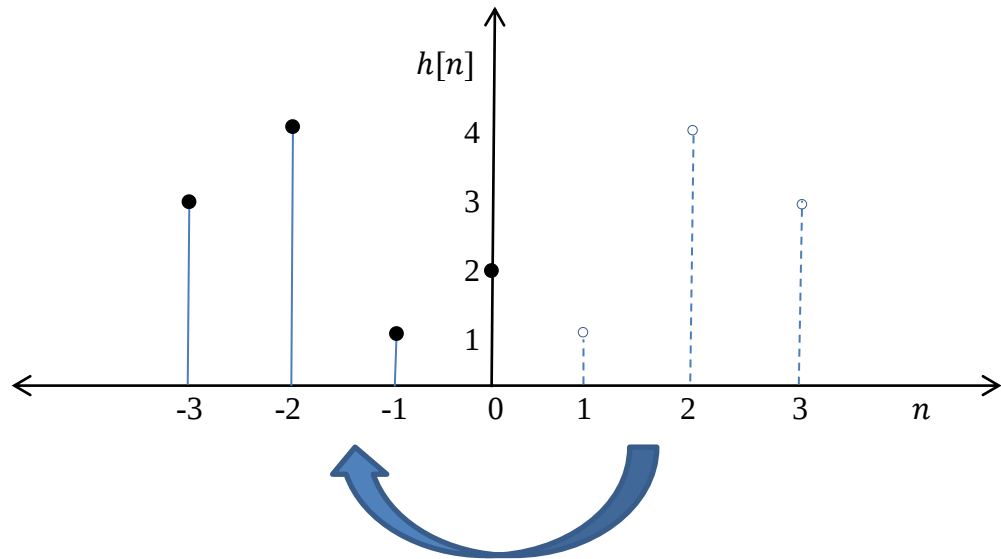


Figure 7.2

The following Figures 7.3 to 7.9 represent the visual construction to our detailed discrete calculations. One can readily see that as the plots progress the result of the sequential products and sums become apparent.

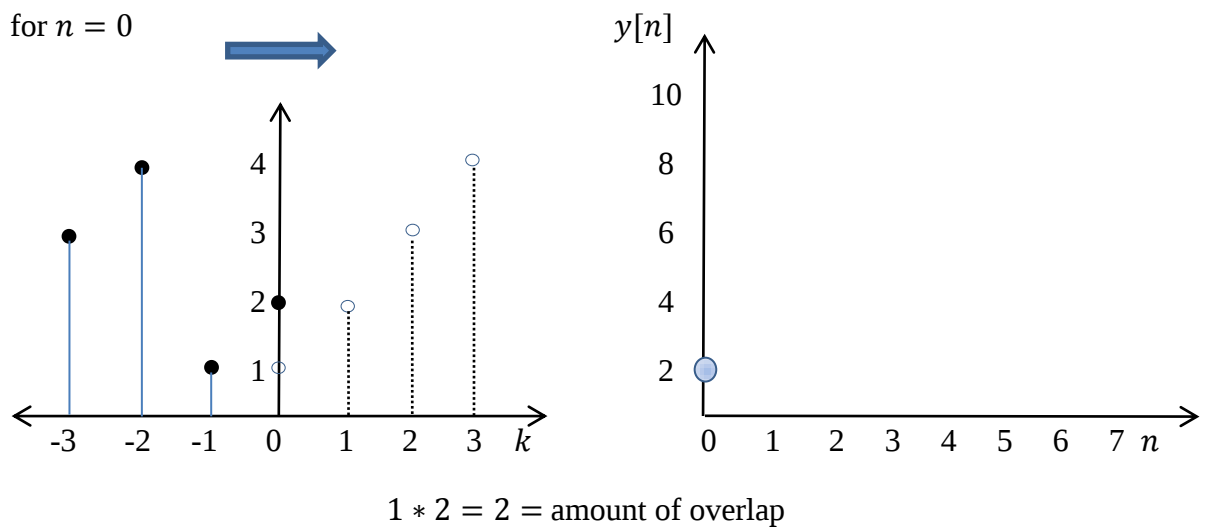


Figure 7.3

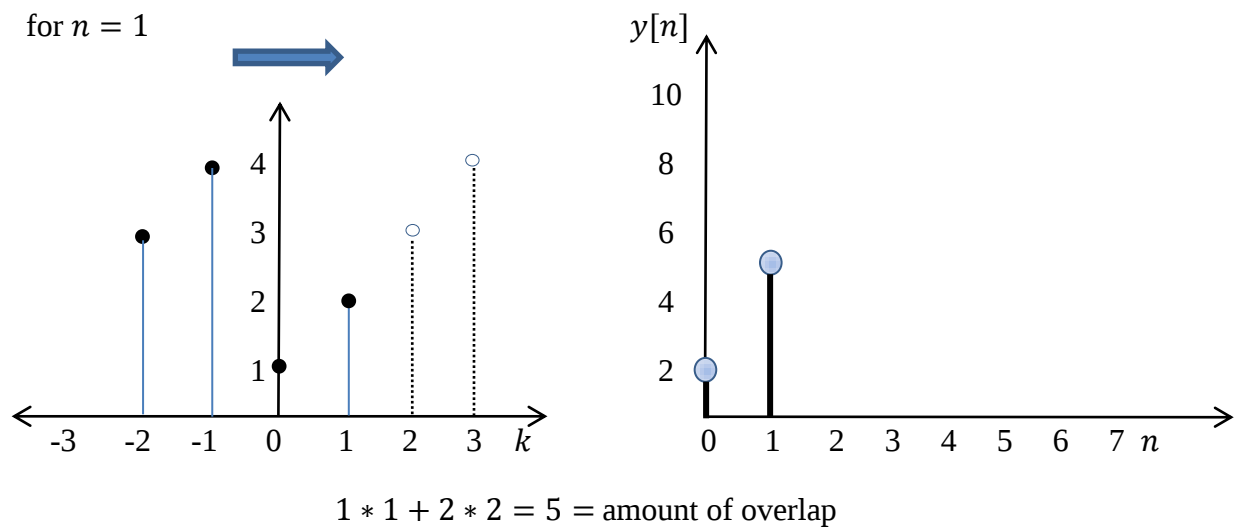


Figure 7.4

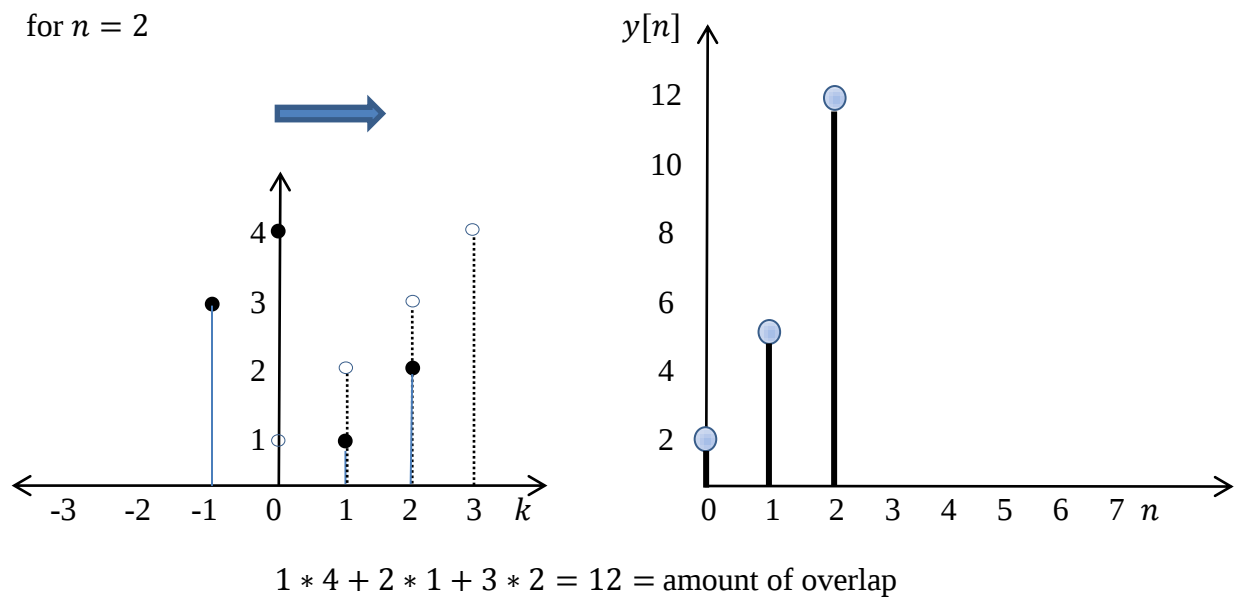


Figure 7.5

for $n = 3$

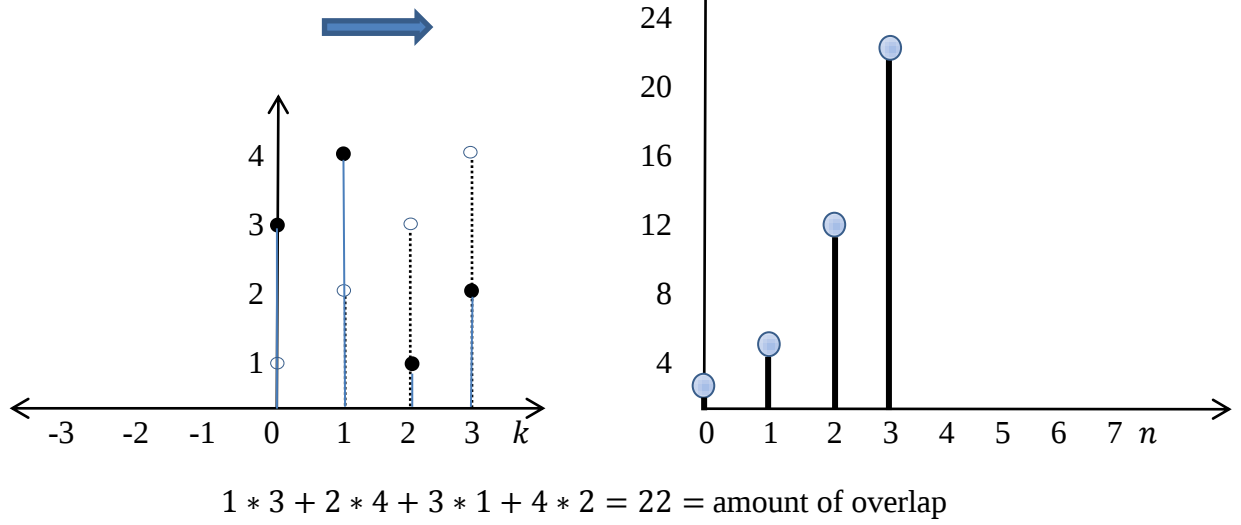


Figure 7.6

for $n = 4$

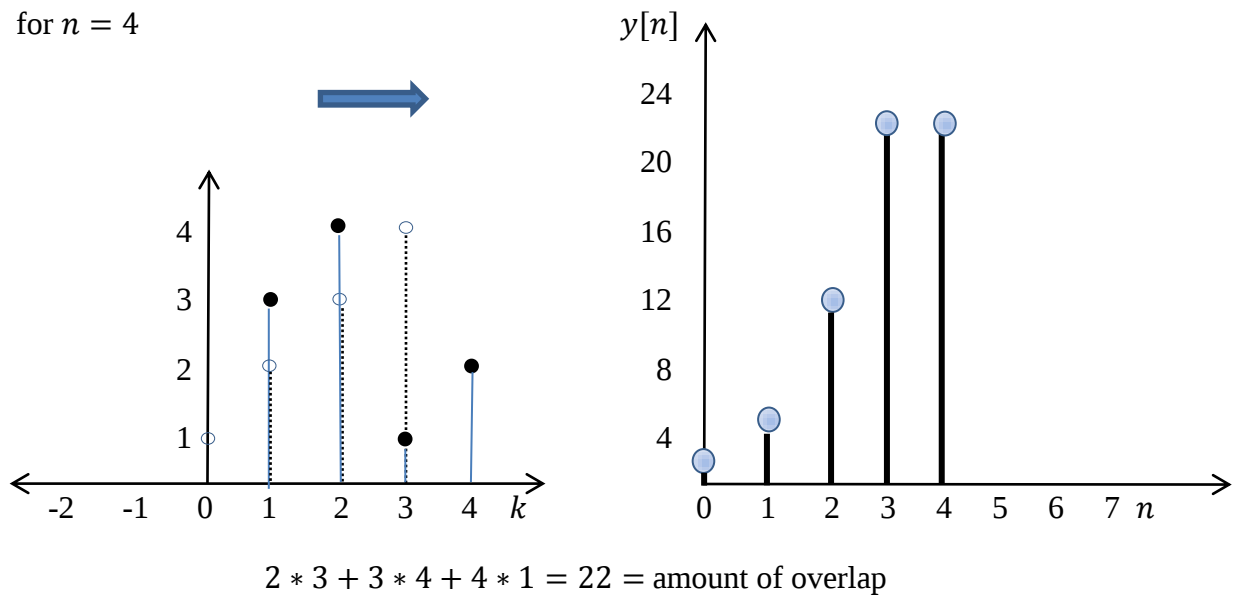


Figure 7.7

for $n = 5$

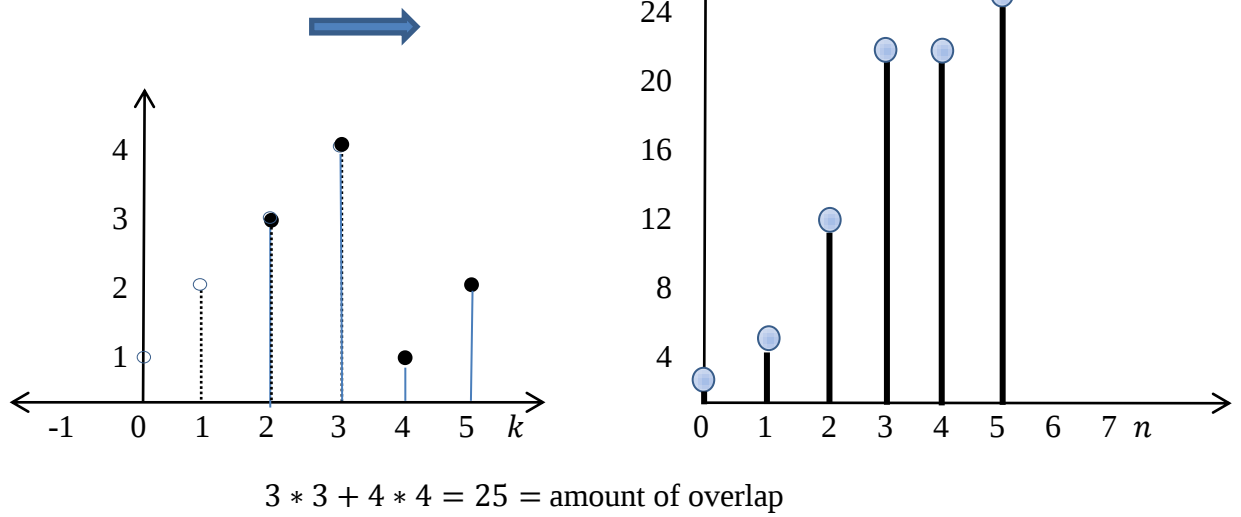


Figure 7.8

for $n = 6$

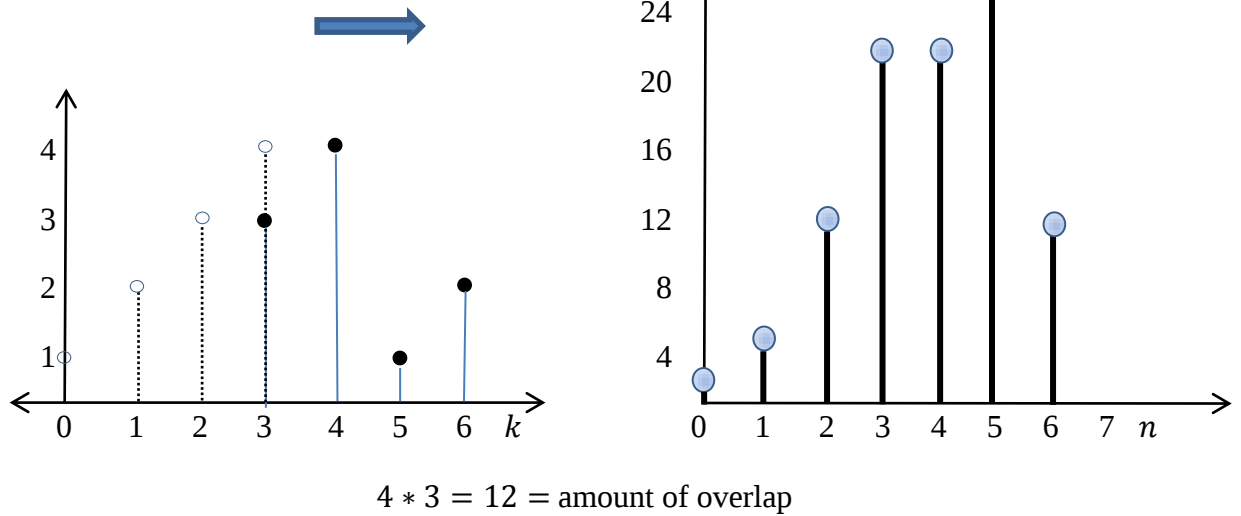


Figure 7.9

Notice $y[5]$ in Figure 7.9 and our historical discussion in Section 3.2. The final output signal can reveal much information about the entire system. For our discussion of the cross-ambiguity function, the peak in our convolution result tells us much about what components constitute this particular signal sample. The peak represents an optimum combination of events in our case revealing the specific time shift involved in this peak.

8. Continuous-Time Convolution

This discrete case extends to continuous intervals by the real convolution integral. We now present the continuous convolution operation in its familiar form [5]

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x - u)du$$

where the symbol $*$ denotes the convolution operator, and $f(u)$ and $g(x - u)$ are two independent random quantities. The variable x is a fixed value, and the variable u is the integration variable. As such, the result, $(f * g)(x)$, is influenced by an input, $f(u)$ which was weighted (multiplied) by the quantity $g(x - u)$ and then summed over the entire real number line. However, we will derive the continuous convolution integral from the discrete-time convolution expression.

Note that we can approximate the integral of some function $s(u)$ by its discrete counterpart

$$\int_a^b s(u)du = \lim_{n \rightarrow \infty} \sum_{i=1}^n s(u_i)\Delta u$$

where Δu represents the width of the approximating rectangle [37]. To this end the following illustration, Figure 8.1, is presented [28] and [39].

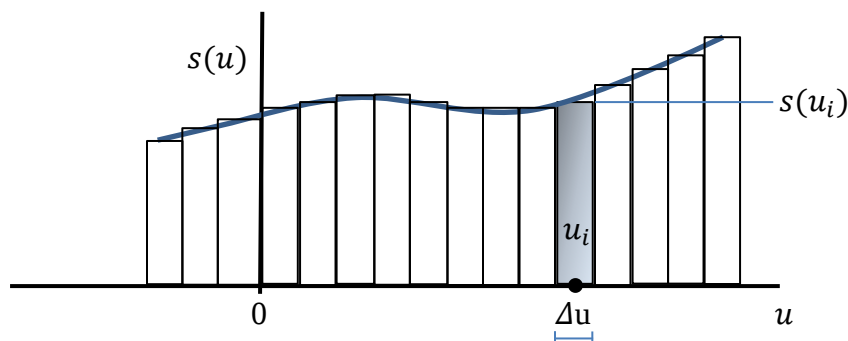


Figure 8.1

We see that we can approximate the area under the curve which represents the continuous signal function $s(u)$ by rectangles. As we increase the number of rectangles the approximation becomes more precise. At this point, we reintroduce the right hand side to be our discrete time convolution formula.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Letting $s(u_i) = x[k_i]h[n-k_i]$ and letting $\Delta u = \Delta k$ we can rewrite the discrete time convolution formula as a limit [3] and [28].

$$\lim_{m \rightarrow \infty} \sum_{i=1}^m x[k_i]h[n-k_i] \Delta k$$

As we take the limit, this yields our real convolution integral [28]

$$\int_{-\infty}^{\infty} x(k)h(n-k)dk$$

8.1 Properties of Convolution

Convolution has many important properties that ultimately affect how the cross-ambiguity function is defined. One of the properties of convolution is that it is commutative [16], [17], [27] and [33],

$$f * g = g * f,$$

or this property may also be written as

$$\int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} g(u)f(x-u)du$$

The commutative property accounts for why the convolution integral is written differently from author-to-author. The commutative property states that the order in which the functions occur in the operation do not change the result. As stated in [35], the convolution integral enjoys this property as a result of it being reversible. We will discuss later, and in more depth, this idea of reversibility.

The second property of convolution is that it is associative [4], [16], [17] and [27] such that,

$$f * (g * h) = (f * g) * h$$

The associative property states that the functions being convolved may be grouped differently if their sequence remains unchanged. The third property of convolution is that it is distributive [4], [16] and [27],

$$f * (g + h) = f * g + f * h$$

The distributive property states that multiplication of the function can be handed over addition. The fourth property of convolution is its impulse property [31],

$$f * \delta = f$$

This property states that when a signal is convolved with the Dirac function we retain the original signal function. The final property of convolution is multilinearity [30],

$$a(f * g) = (af) * g = f * (ag)$$

These properties represent the set of algebraic functions of convolution called C^* -algebra. With these basic properties defined we can proceed to an analysis of continuous convolution using the following RCI as our standard form.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$$

Where the function, f , will be one signal, and g , will be another signal. The integration variable is t , and τ will provide for the indexing through the interval. We switch to these variables since as we approach discussing the cross-ambiguity function specifically, time will become an even more important factor in describing its functions. In sum, the space of functions together with the convolution operator is an algebraic space.

8.2 Continuous and Discrete Convolution Compared

Given our extensive discussion of discrete-time convolution, the continuous convolution offers a direct parallel to the basic concepts presented in the discrete case. First and foremost, of course, our RCI performs its superposition across now an **infinite** number of overlapping combinations. Secondly, the upper and lower indices, as discussed in Section 4.1, translate well into the limits of integration. In our case the

lower limit of integration is determined as the $\text{MAX}\{\text{left limit of } f(\tau), \text{left limit of } g(t - \tau)\}$ and the upper limit of integration is determined as $\text{MIN}\{\text{right limit of } f(\tau), \text{right limit of } g(t - \tau)\}$ [3]. Finally, as discussed in Section 7, the process of Faltung, i.e. folding or flipping of the signal, is preserved.

8.3 Continuous Convolution Exercise

Like the discrete time case in Section 7, the purpose of the following exercise is to illustrate the basic concepts involved in continuous convolution. There will be two different demonstrations using different signals in order to introduce important concepts concerning the cross ambiguity function. Analyzing the shape of the resulting signal is essential in deriving important features to determine information such as detection and location. Thus, we will be interested in items such as peaks, if any, in the resulting signal, clusters of peaks, and the spread around the peaks. The following exercise is limited to the time domain, so our results are bounded in simple two dimensional plots of the data. As we introduce the frequency component into our framework, the discussion will include three dimensional outputs.

8.4 Continuous Convolution Exercise – Signal Set One

The purpose in choosing the particular signal set in this section is to introduce and demonstrate the exponential factor in the convolution process. The cross-ambiguity function makes particular use of the exponential in its calculations albeit principally for the frequency off-set. Furthermore, the exponential term in our case also makes use of “ $-x$ ” as its exponent. In the demonstration we will also introduce the rectangular pulse which this thesis makes its basic signal form in its survey of the cross-ambiguity function.

The first signal of this section’s signal set is the truncated exponential function [5] defined as

$$E(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The graph of this function is given as Figure 8.2.

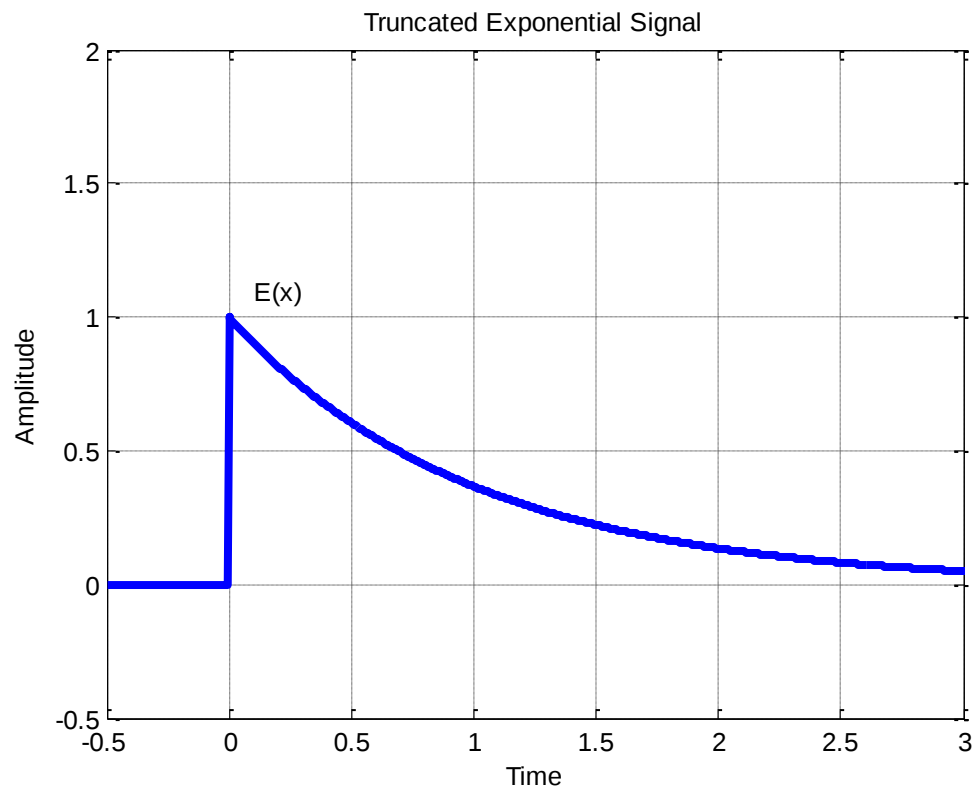


Figure 8.2

The second signal is the rectangular pulse [5] defined as

$$\Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

The graph of this function is given as Figure 8.3.

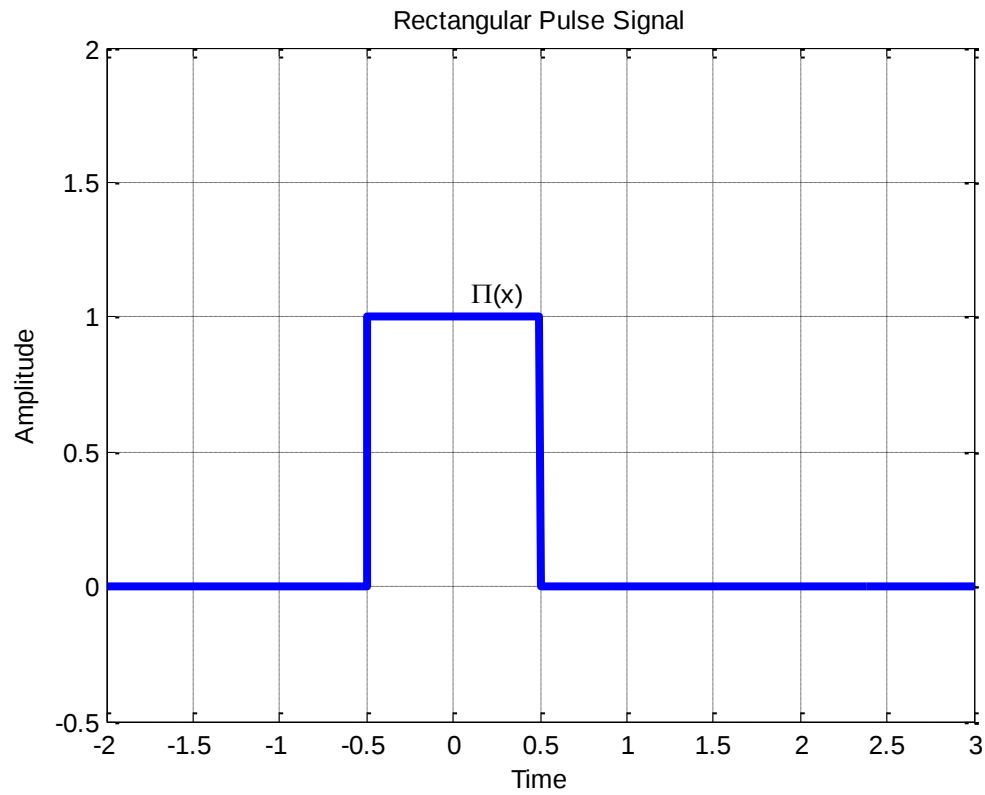


Figure 8.3

MATLAB also defines the rectangular pulse as in [5]. Three cases will now be presented to establish and present the limits of integration, ranges of overlap, and the resulting functions in each case that, when combined - usually in a piecewise fashion, composite the output signal $y[n]$.

Case I: The range for the shift of t is $t < -\frac{1}{2}$

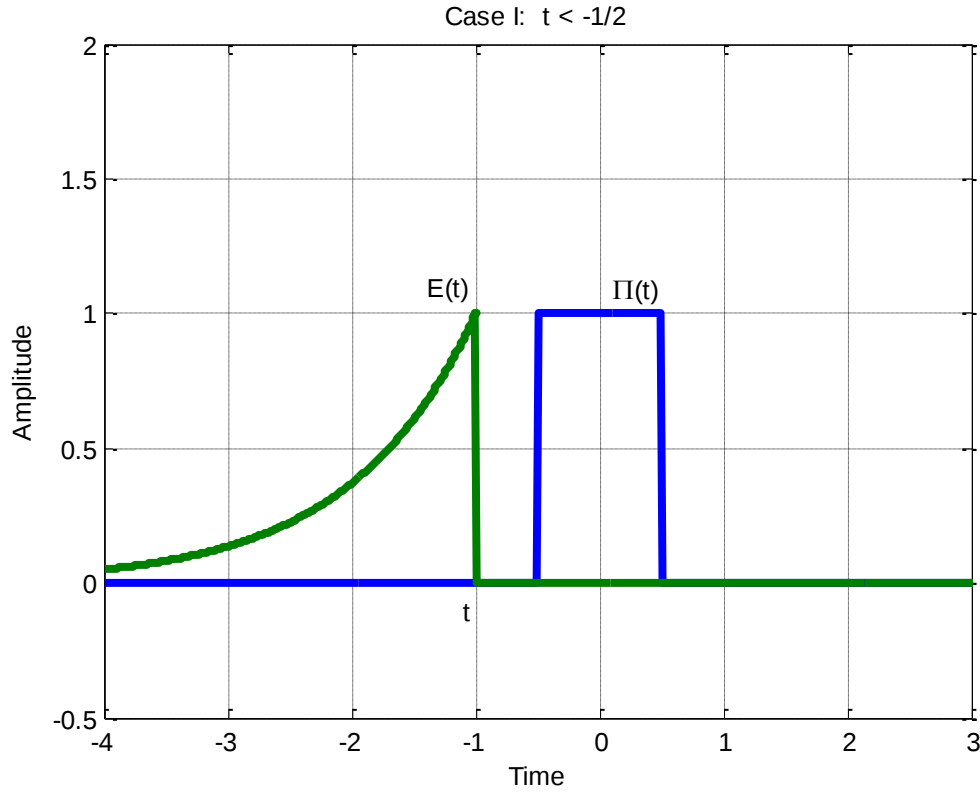


Figure 8.4

Notice in Figure 8.4, the signal functions do not overlap. Since $\Pi(t)$ is defined as zero for this range we have

$$y(t) = (\Pi * E)(t) = \int_{-\infty}^{\infty} \Pi(\tau)E(t - \tau)d\tau =$$

$$\int_{-\infty}^t 0 * e^{-(t-\tau)}d\tau =$$

$$\int_{-\infty}^t 0 * d\tau = 0$$

Note that the definite integral of zero is zero. The purpose of the integral is to find the area under the curve of the convolution function, $\Pi(\tau)E(t - \tau)$. Since the function itself is zero, the corresponding area under this curve is zero. This is not the same as taking the anti-derivative or indefinite integral of the function, 0, per the Fundamental Theorem of Calculus which would yield a constant since the derivative of a constant is zero [22].

Case II: The range for the shift of t is $-\frac{1}{2} \leq t < \frac{1}{2}$

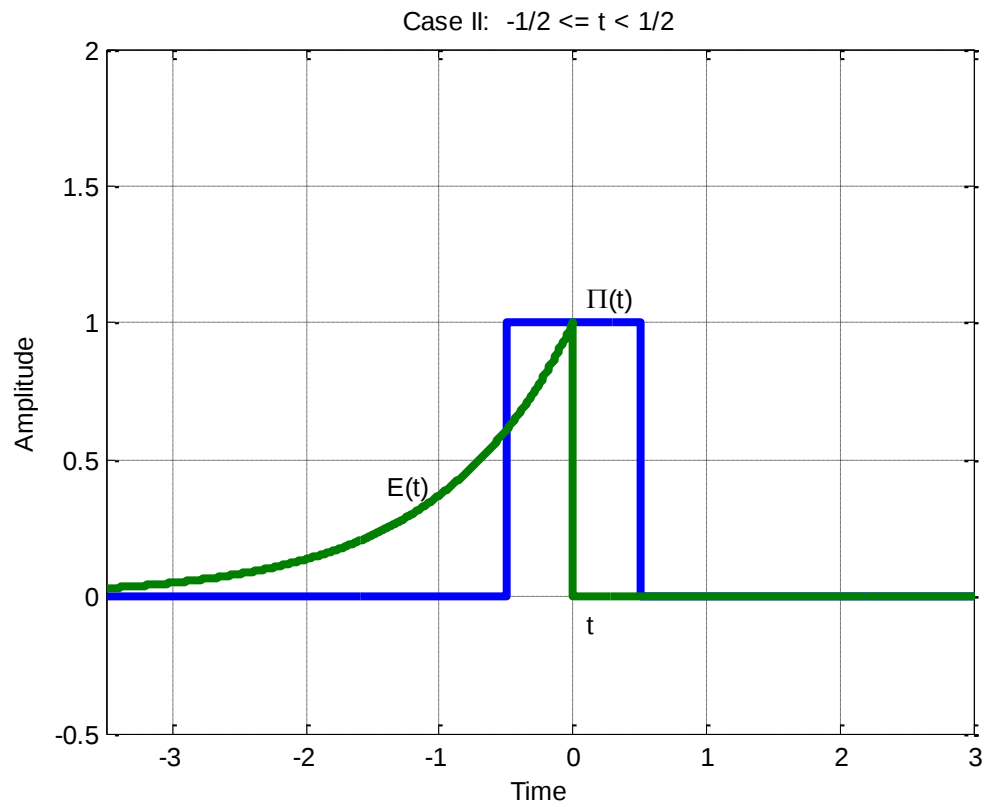


Figure 8.5

Notice in Figure 8.5, the signal functions do overlap. Both functions are defined for this range, so we have

$$y(t) = (\Pi * E)(t) = \int_{-\infty}^{\infty} \Pi(\tau)E(t - \tau)d\tau =$$

$$\int_{-\frac{1}{2}}^t 1 \cdot e^{-(t-\tau)}d\tau =$$

$$\int_{-\frac{1}{2}}^t 1 \cdot e^{\tau-t}d\tau =$$

$$\int_{-\frac{1}{2}}^t e^{\tau-t}d\tau =$$

$$\begin{aligned}
 \int_{-\frac{1}{2}}^t e^{\tau} e^{-t} d\tau &= \\
 e^{-t} \int_{-\frac{1}{2}}^t e^{\tau} d\tau &= \\
 e^{-t} e^t - e^{-t} e^{-\frac{1}{2}} &= \\
 e^{t-t} - e^{-t-\frac{1}{2}} &= \\
 e^0 - e^{-t-\frac{1}{2}} &= \\
 1 - e^{-t-\frac{1}{2}} &=
 \end{aligned}$$

As we shall see from the output from convolution graph, the amount of overlap will reach a peak at the upper limits of integration.

Case III: The range for the shift of t is $\frac{1}{2} \leq t$

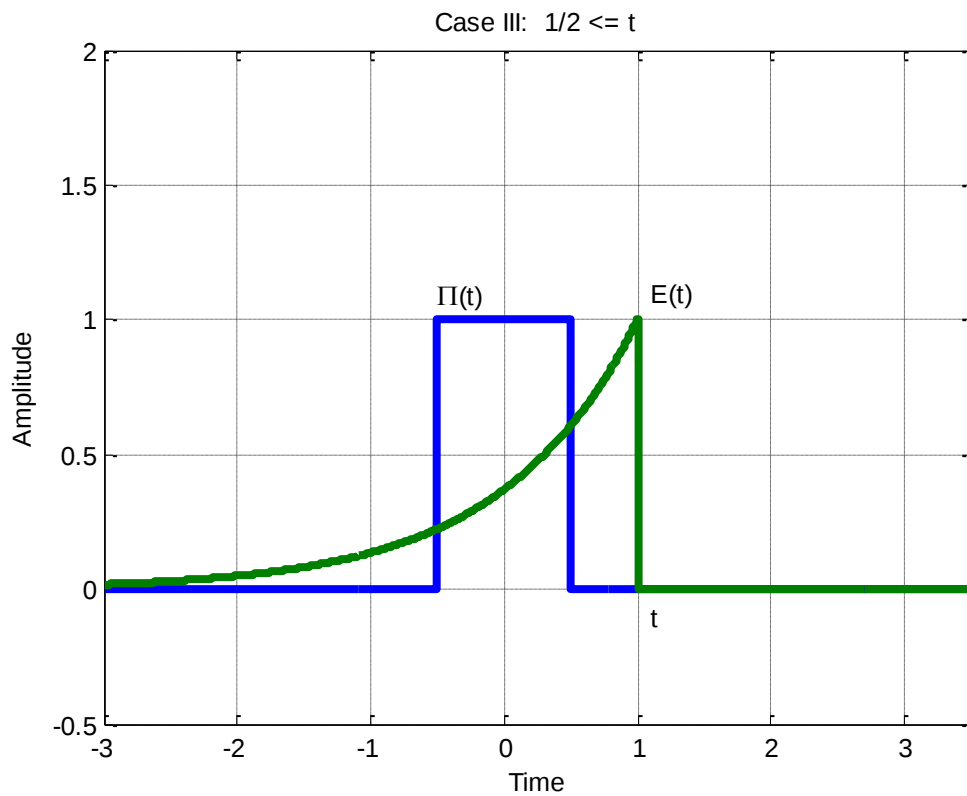


Figure 8.6

Notice in Figure 8.6, the signal functions do overlap. $E(t)$ extends to $-\infty$ and so overlaps across the width of the rectangular pulse, $\Pi(\tau)$. So the limits of integration are $-\frac{1}{2} \leq t \leq \frac{1}{2}$. Both functions are defined for this overlap. We have the same integration steps as in Case II, however, the resulting function is now evaluated at Case III's particular limits of integration.

$$\begin{aligned} e^{-t}e^{\frac{1}{2}} - e^{-t}e^{-\frac{1}{2}} &= \\ e^{-t}\left(e^{\frac{1}{2}} - e^{-\frac{1}{2}}\right) &\cong \\ 1.0422e^{-t} \end{aligned}$$

Notice from the output of convolution, that as t extends to $+\infty$, the amount of overlap within the limits of integration tapers-off significantly from the peak of integration identified in Case II.

The Output Signal - $y(t)$

Combining the resulting convolution products from each of the three aforementioned cases defines the output signal across all ranges. The resulting piecewise function is

$$(\Pi * E)(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 - e^{-t-\frac{1}{2}} & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 1.0422e^{-t} & \frac{1}{2} < t \end{cases}$$

Figure 8.7 represents the piecewise signal function. In the real world, the signal would not be infinitely long to the right. Given power concerns and the few, if any, applications requiring a pseudo-infinitely long transmission, the signal would be terminated at some time, t .

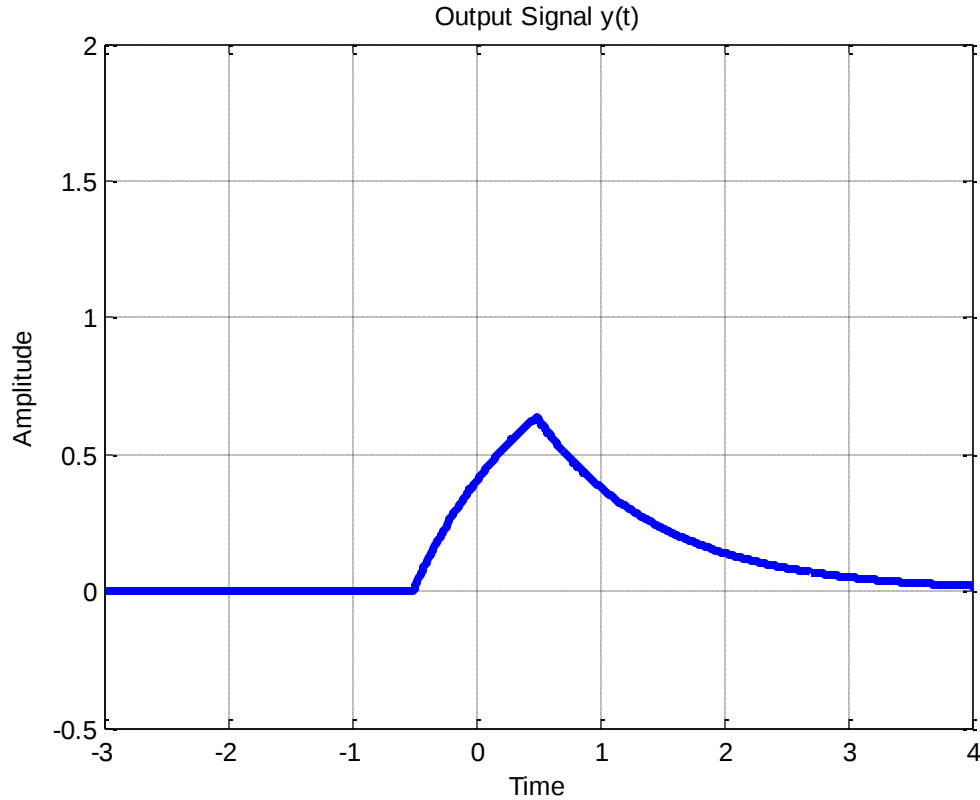


Figure 8.7

The MATLAB code is provided in Appendix One and is a heavily modified version of [2].

8.5 Continuous Convolution Exercise – Signal Set Two

The purpose in choosing the particular signal set in this section is to introduce and demonstrate a real-world model of two time domain rectangular pulses convolved into one output signal. This thesis will then build upon this model of two rectangular radar pulses by introducing it into cross-ambiguity function calculations as well as take a particular real world signal and convolve it. The cross-ambiguity function is used frequently in radar detection, location and characterization applications. Both rectangular pulse functions $x(t)$ and $h(t)$ in this example are defined as $\Pi(x)$. The purpose of this is to also introduce a set of concepts which will be explained in detail later in the thesis. These concepts involve various types of correlation, such as auto-correlation which this example partially illustrates.

Case I: The range for the shift of t is $t < -\frac{1}{2}$

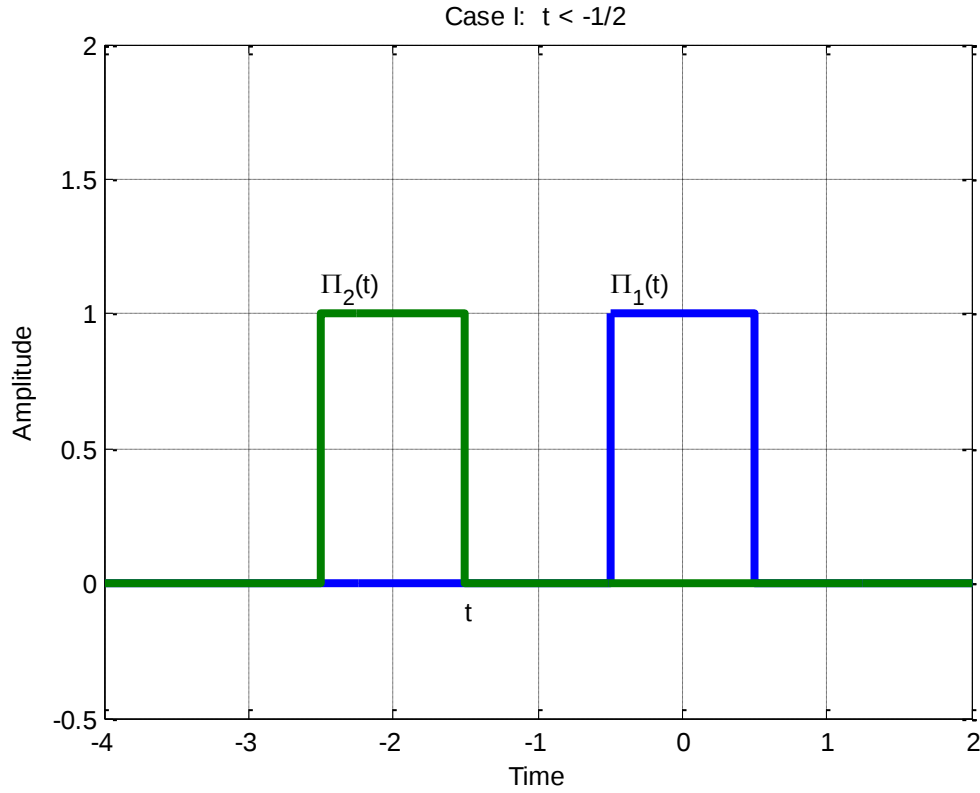


Figure 8.8

Notice in Figure 8.8, the signal functions do not overlap. $\Pi_x(t)$ is not defined over this range, and for the reasons cited in Case I of the previous section, the convolution is zero.

$$y(t) = (\Pi_x * \Pi_h)(t) = \int_{-\infty}^{\infty} \Pi_x(\tau) \Pi_h(t - \tau) d\tau = 0$$

Case II: The range for the shift of t is $-\frac{1}{2} \leq t < \frac{1}{2}$

Notice for this Case II, the impulse signal function is defined at a specific time. It does not extend to $-\infty$ as the truncated pulse function did in Signal Set One.

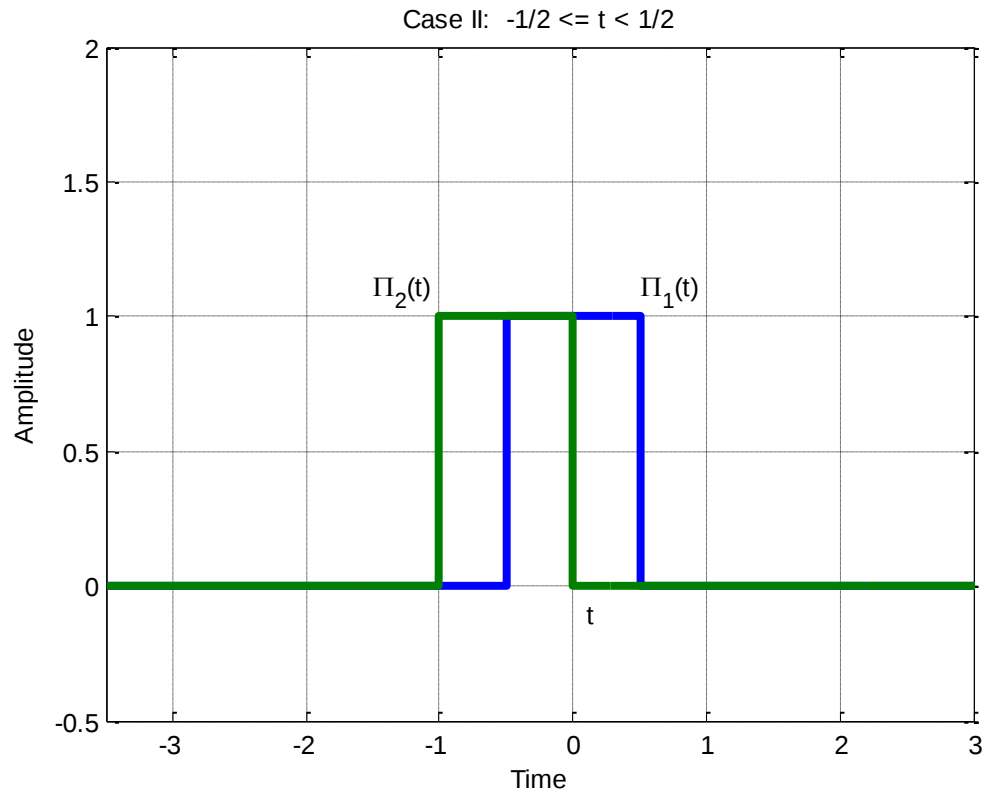


Figure 8.9

Notice in Figure 8.9, the signal functions do overlap. The limits of integration are from $-\frac{1}{2}$ to t , so we have

$$y(t) = (\Pi_x * \Pi_h)(t) = \int_{-\infty}^{\infty} \Pi_x(\tau) \Pi_h(t - \tau) d\tau =$$

$$\int_{-\frac{1}{2}}^t 1 \cdot 1 d\tau =$$

$$\int_{-\frac{1}{2}}^t 1 d\tau =$$

$$t - \left(-\frac{1}{2}\right) =$$

$$t + \frac{1}{2}$$

Case III: The range for the shift of t is $\frac{1}{2} \leq t < \frac{3}{2}$

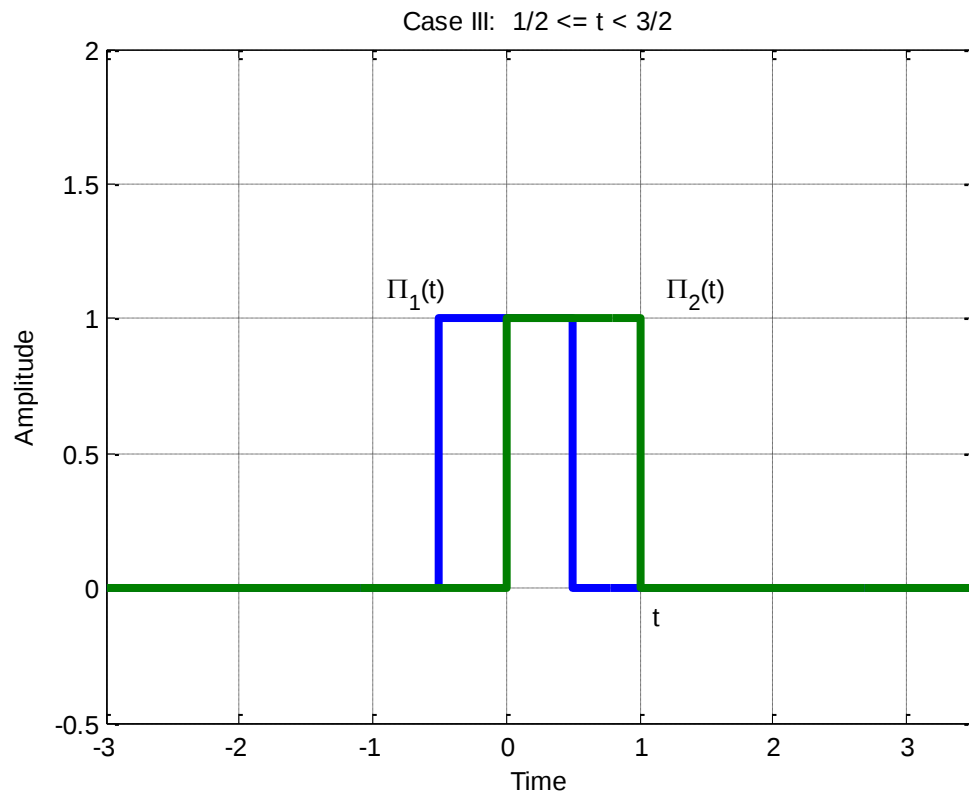


Figure 8.10

Notice in Figure 8.10, the signal functions do overlap. The limits of integration are from $t - 1$ to $\frac{1}{2}$, so we have

$$y(t) = (\Pi_x * \Pi_h)(t) = \int_{-\infty}^{\infty} \Pi_x(\tau) \Pi_h(t - \tau) d\tau =$$

$$\int_{t-1}^{\frac{1}{2}} 1 \cdot 1 d\tau =$$

$$\int_{t-1}^{\frac{1}{2}} 1 d\tau =$$

$$\frac{1}{2} - (t - 1) =$$

$$\frac{3}{2} - t$$

As we shall see from the output from convolution graph, the amount of overlap will reach a peak at the upper limits of integration.

Case IV: The range for the shift of t is $\frac{3}{2} \leq t$

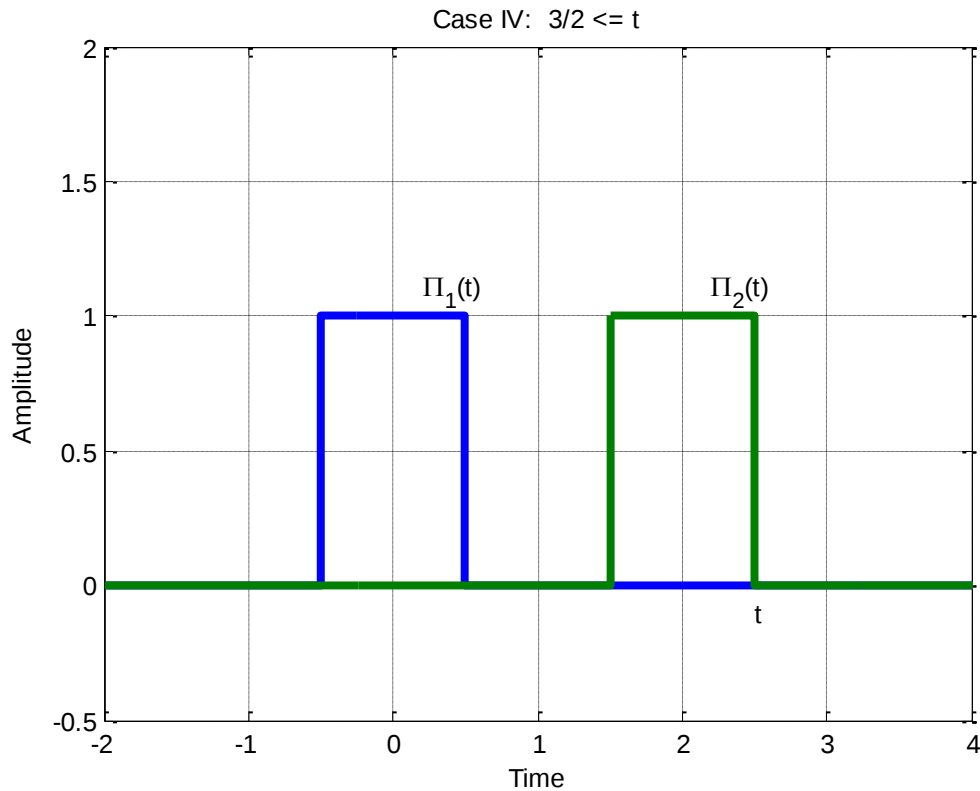


Figure 8.11

Notice in Figure 8.11, the signal functions do overlap, and their convolution product is zero.

The Output Signal - $y(t)$

Combining the resulting convolution products from each of the four aforementioned cases defines the output signal across all ranges. The resulting piecewise function is

$$y(t) = (\Pi_x * \Pi_h)(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ t + \frac{1}{2} & -\frac{1}{2} \leq t < \frac{1}{2} \\ \frac{3}{2} - t & \frac{1}{2} \leq t < \frac{3}{2} \\ 0 & \frac{3}{2} \leq t \end{cases}$$

Figure 8.12 illustrates the piecewise signal function. In the real world, the signal would, like the input signals, be somewhat rounded at the “kinks” since all transmission devices have small rise and fall times as they produce the power to generate the signal.

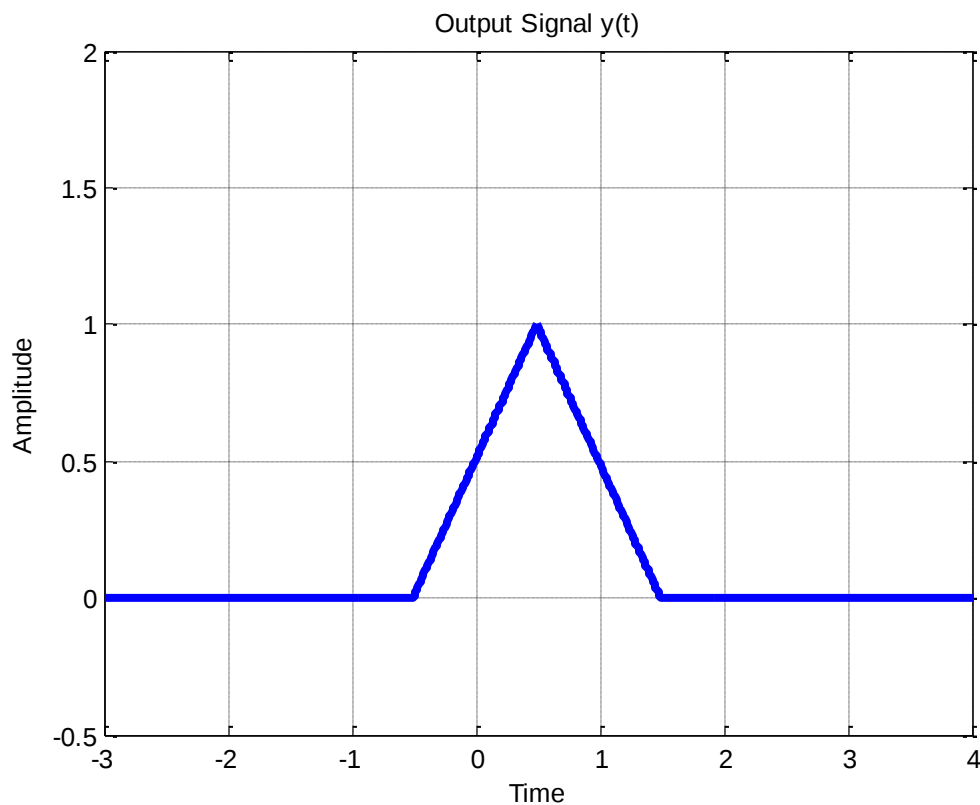


Figure 8.12

The most important feature, like that of Signal Set One, is the central peak formed by the convolution. Notice the peak we can determine the time of the peak by looking at the graph and see that it is $\frac{1}{2}$. By inspecting such features we can draw direct inferences about the possible location of the signal. The cross-ambiguity function uses just this sort of time domain inference when calculating two pulse radar signals presumably from the

same emitter. The MATLAB code for this convolution is provided in Appendix One and is a heavily modified version of [2].

8.6 Summary

Discrete and continuous time convolutions have been discussed and illustrated in mathematical depth. The main mathematical feature developed has been the idea of finding a product and summing the results over a defined range producing a series of magnitudes in the time domain. These magnitudes appear to have a pattern chiefly among the peak at which they converge. MATLAB has also been introduced to develop these concepts and will continue to be used. The concept that will be used throughout the thesis in modeling further environs is the processing of two, apparently similar and possibly the same, pulses from an emitter. The next section extends to the next dimension, the frequency domain, of the discussion further building in the convolution of not only a time component, but frequency one as well.

9. The Frequency Dimension

The time domain convolution aspects of the cross-ambiguity function have been thoroughly explored. The sum product of the magnitude of individual signals samples between two time-shifted signals has been demonstrated to yield an output signal. There is another aspect of the cross-ambiguity function that exploits another physical feature in our signal set in addition to the time delay, τ , and that is, ν , the frequency off-set or Doppler shift. The frequency off-set is contained within the frequency domain of the signal set. Notice in the cross-ambiguity function [36] below

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g^*(t + \tau)e^{-j\omega\nu t}d\tau$$

there is an exponential factor $e^{-j\omega\nu t}$ and j is the imaginary unit $\sqrt{-1}$. The first response upon inspecting the formula is, “Where did it come from?”. The response lies within the frequency domain and how signals in that domain are represented. This thesis will explain it mathematically.

9.1 Frequency Off-set

The frequency off-set variable represents the Doppler shift. The Doppler shift is the measured frequency change between a moving transmitter or receiver, or both. A stationary transmitter, for example, transmits a signal on a certain frequency. A moving receiver will perceive that transmission at a slightly different frequency due to wave

compression or elongation. Figure 9.1 depicts the notion of a receiver moving away from a Stationary Transmitter.

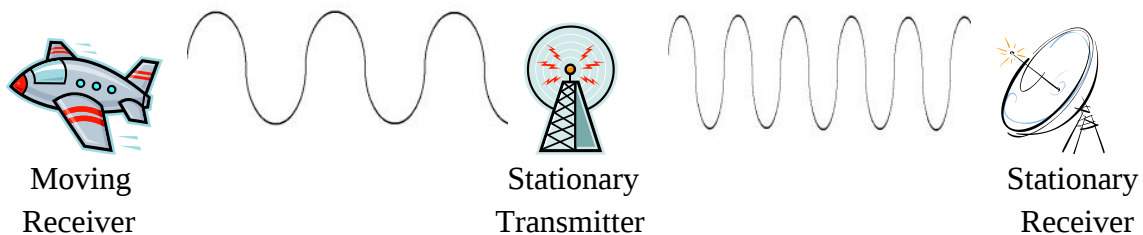


Figure 9.1

The Stationary Transmitter emits a signal at a certain frequency. The Stationary Receiver will receive the signal at the same frequency since there is no relative movement between them. The Moving Receiver is moving away from the Stationary Transmitter at a certain constant speed. The Moving Receiver perceives the captured frequency at a slightly different frequency because of relative motion between the two. In this particular case, the Moving Receiver is moving away from the Stationary Transmitter so the frequency at which the Moving Receiver collects the signal is lower than the actual transmitted frequency. This is because the Moving Receiver's additional radial velocity away from the Stationary Transmitter induces a longer perceived wavelength and therefore lower frequency at the Moving Receiver. If the Moving Receiver was moving toward the Transmitter it would likewise perceive the signal at a slightly higher frequency. The Doppler Effect is a well-known physical feature and can be calculated quite easily for the Moving Receiver which is traversing away from the Stationary Transmitter

$$\Delta f = -f_{ST} \frac{v_{MR}}{c}$$

where v_{MR} is the frequency observed at the Moving Receiver, f_{ST} is the transmitted frequency and c is the speed of light, 299,792,458 m/s. Assume, for example, $f_{ST} = 9,345,000$ Hertz and $v_{MR} = 13.4$ m/s. The change in frequency or frequency off-set, $\Delta f = -0.417698$ Hertz, almost half a cycle. While a relatively slight decrease, the timing systems on board receivers are highly sensitive to these small changes and can register them with great accuracy. Thus, ν , in the cross-ambiguity function can be effectively measured and calculated in an overall product.

9.2 Euler's Identity

One of the fundamental building blocks of signal processing is Euler's identity which defines a relationship between the complex plane and sinusoids. The identity can be used to express transformations of signals in both the time and frequency domains for many different purposes. The Euler identity is extremely useful not only in its profound implications for expressing real signals using the complex plane, but its elegance and compactness make it computationally easy to use. The Euler identities [37], [34] are

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta)$$

where $j = \sqrt{-1}$ and θ is the angle between the vector in the complex plane and the real axis. A brief description of the complex plane and the derivation of Euler's identity is instructive since it is key in understanding the calculations of the cross-ambiguity function.

9.3 The Complex Plane

Complex numbers are important components in describing signals and signal processing systems. They form not only the basis for the signal, but also mixing and analysis of these signals in complex signal processing systems. It begins with a description of a complex number and its illustration in the complex plane. The complex number, z , has two parts: a real part, a , and an imaginary part, b [37].

$$z = a + jb$$

We can further express the real and imaginary parts as $Re(z) = a$ and $Im(z) = jb$ as is frequently the case in the literature. This relationship [7] can be plotted as Figure 9.2.

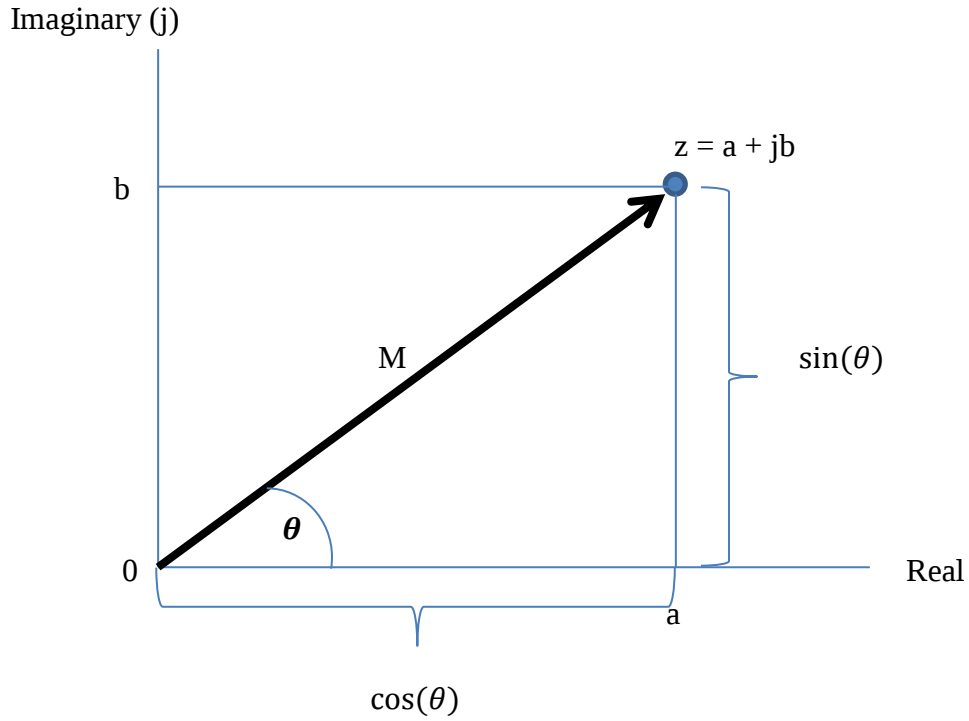


Figure 9.2

There are several points about Figure 9.2 that are important to our discussion of the cross-ambiguity function. The point, z , is not just a point, but can be looked upon as expressing a vector with magnitude, M . M will come to represent an important element in the signal function. Here it is a convenient illustration for a magnitude, but in later sections it will become the signal envelope of a modulated signal, such as, $\Pi(t)$. For now, as the excellent reference [23] instructs, we will call this a phasor in the both the mathematical and communications vernacular. The phasor can rotate in either a counter-clockwise or clockwise direction denoted by the imaginary unit, $+j$ or $-j$, respectively. In describing the 90 degree phase shift of the cosine sinusoid, these imaginary operators are employed and a rotation table is described in Table 9.1 for $\theta = \pm j$. Further discussion of phase shifting these signals is described in Section 9.6, Hilbert Transform.

	j	j^2	j^3	j^4
$\cos(\theta)$	$\sin(\theta)$	$-\cos(\theta)$	$-\sin(\theta)$	$\cos(\theta)$

	$(-j)$	$(-j)^2$	$(-j)^3$	$(-j)^4$
$\cos(\theta)$	$-\sin(\theta)$	$-\cos(\theta)$	$\sin(\theta)$	$\cos(\theta)$

Table 9.1

The rectangular form is the most usual form encountered. The trigonometric form can be easily computed by noticing

$$a = M\cos(\theta) \quad \text{and} \quad b = M\sin(\theta) \quad \text{so,}$$

$$z = M\cos(\theta) + jM\sin(\theta) \quad \text{or,}$$

$$z = M[\cos(\theta) + j\sin(\theta)]$$

The purpose here is to introduce the cosine and sine terms which are the traditional functions representing not just simple sinusoidal signals, but are the basic components of more complicated modulations.

The trigonometric form is used frequently in signal processing as the quadrature description of a signal and the polar (exponential) form is also used frequently in signal processing. Note the $e^{j\theta}$ term. This is similar to the term demonstrated in the continuous convolution exercise for the truncated exponential and is also the term encountered in the cross-ambiguity function. The polar form is used quite frequently given its elegant and ease of use in mathematics. It is also used to describe, what this theses will present later, the analytic signal used in the cross-ambiguity function.

9.4 Euler Identity Derivation

The Euler identities can be proved through the use of the Taylor Series polynomials. We start with the basic exponential form [37]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad -\infty < x < \infty$$

and put $x = j\theta$ so,

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{j\theta^n}{n!} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \dots, \quad -\infty < x < \infty$$

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{j\theta^n}{n!} = 1 + \frac{j\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \dots, \quad -\infty < x < \infty$$

The alternating terms in the sequence reduce to the Taylor Series representations [37] for cosine and sine, since

$$\sin(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = j\theta - j \frac{\theta^3}{3!} + j \frac{\theta^5}{5!} + \dots$$

$$\cos(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$$

So the resulting Euler Identity is defined

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{j\theta^n}{n!} = \cos(\theta) + j\sin(\theta)$$

where

$$e^{-j\theta} = \sum_{n=0}^{\infty} \frac{-j\theta^n}{n!} = \cos(\theta) - j\sin(\theta)$$

is the negative (complex conjugate) of the basic identity. By the setting these two formulas equal to each other we have

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

The significance of this result cannot be overstated [23]. Real quantities, the $\cos(\theta)$ and $\sin(\theta)$, can be expressed in complex exponential terms. Taking the basic identities for the $\cos(\theta)$ and $\sin(\theta)$ established in the equations immediately above, we substitute in the frequency and time domains

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

$$\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

Like our previous statement, we can now extend our submission to the fact that *real valued signals* can be expressed as complex exponentials. These real world signals are comprised of these basic sine and cosine forms from not only simple sinusoidal signals, but more complicated modulations like the rectangular pulse which, as we will see, is comprised of the summation of many frequency components. As such, the frequency domain will now be explored with detail.

9.5 The Frequency Transform

The frequency domain of a time domain signal can be expressed using the following frequency transform function – called the Fourier Transform [5], [18]

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$$

where $x(t)$ is the time domain representation of a signal, and f is the frequency. As an example, find $F\{x(t)\}$ for

$$x(t) = \cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

using the aforementioned Euler identity for cosine. Plugging into the Fourier Transform and integrating, we have

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} e^{-j2\pi ft} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) e^{-j2\pi ft} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi ft} + e^{-j2\pi f_0 t} e^{-j2\pi ft} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\pi f_0 t - j2\pi ft} + e^{-j2\pi f_0 t - j2\pi ft} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\pi t(f_0-f)} + e^{j2\pi t(-f_0-f)} dt \\
&= \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\pi t(f_0-f)} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\pi t(-f_0-f)} dt
\end{aligned}$$

To simplify, notice the Fourier Transform of the Dirac Function [20] is

$$\begin{aligned}
\frac{2\pi\delta(k)}{|a|} &= \int_{-\infty}^{\infty} e^{jakt} dt \\
&= \frac{1}{2}\delta(f_0 - f) + \frac{1}{2}\delta(-f_0 - f) \\
&= \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)
\end{aligned}$$

Since the Dirac function is an even function [20], i.e.

$$\delta(k) = \delta(-k)$$

The result can be illustrated in Figure 9.3 where two individual frequency components are registered at frequencies f_0 and $-f_0$ with a magnitude of $\frac{1}{2}$.

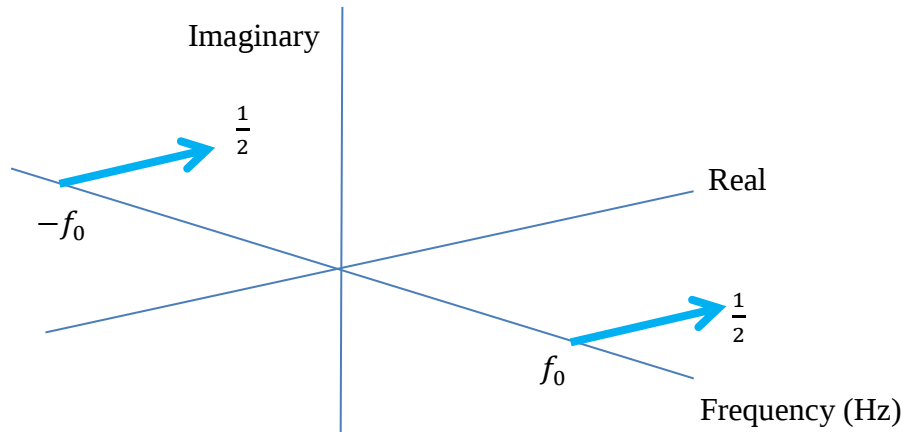


Figure 9.3

We extend the same computation to the sine function using the same Fourier Transform.

$$x(t) = \sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

using the aforementioned Euler identity for cosine. Plugging into the Fourier Transform and integrating, we have

$$\begin{aligned} F\{x(t)\} &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} e^{-j2\pi f t} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} (e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) e^{-j2\pi f t} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} - e^{-j2\pi f_0 t} e^{-j2\pi f t} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} e^{j2\pi f_0 t - j2\pi f t} - e^{-j2\pi f_0 t - j2\pi f t} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} e^{j2\pi t(f_0 - f)} - e^{j2\pi t(-f_0 - f)} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} e^{j2\pi t(f_0 - f)} dt - \frac{1}{2j} \int_{-\infty}^{\infty} e^{j2\pi t(-f_0 - f)} dt \end{aligned}$$

Notice, again, the Fourier Transform of the Dirac Function is

$$\frac{2\pi\delta(k)}{|a|} = \int_{-\infty}^{\infty} e^{jakt} dt$$

which is used again to simplify the subject equation.

$$\begin{aligned} &= \frac{1}{2j} \delta(f_0 - f) - \frac{1}{2j} \delta(-f_0 - f) \\ &= \frac{1}{2j} \delta(f - f_0) - \frac{1}{2j} \delta(f + f_0) \end{aligned}$$

Also, again, notice that the Dirac function is an even function.

Additionally, rationalizing the denominator using

$$1 = -j^2$$

we have

$$= \frac{-j}{2} \delta(f - f_0) + \frac{j}{2} \delta(f + f_0)$$

The result, like the cosine, can be illustrated as Figure 9.4.

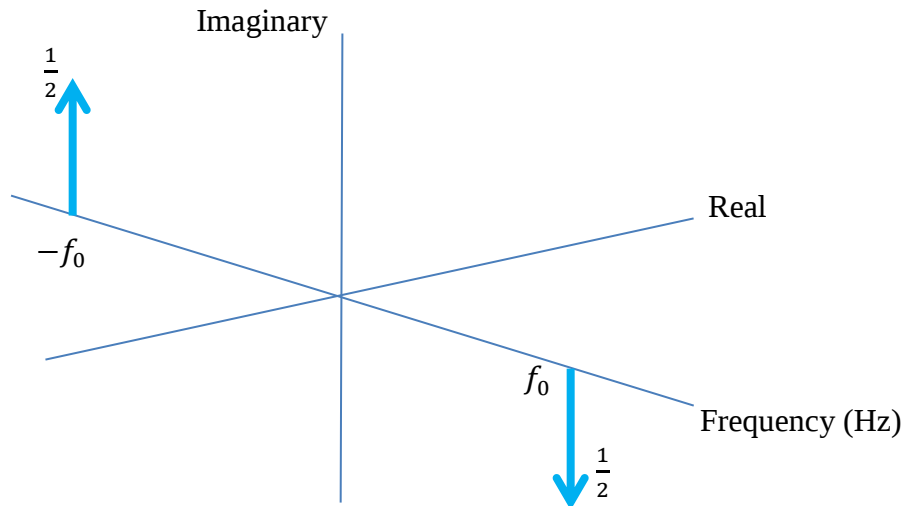


Figure 9.4

Notice, much like the cosine illustration in Figure 9.3, there are two frequency components. However, these are 90° out-of-phase from the cosine components. Of importance, the above Fourier Transform process has produced both positive and negative frequencies – each symmetric about the power axis. When the subject of the analytic signal is discussed, the negative frequency component will be suppressed using the Hilbert Transform. This is key to developing the cross-ambiguity function.

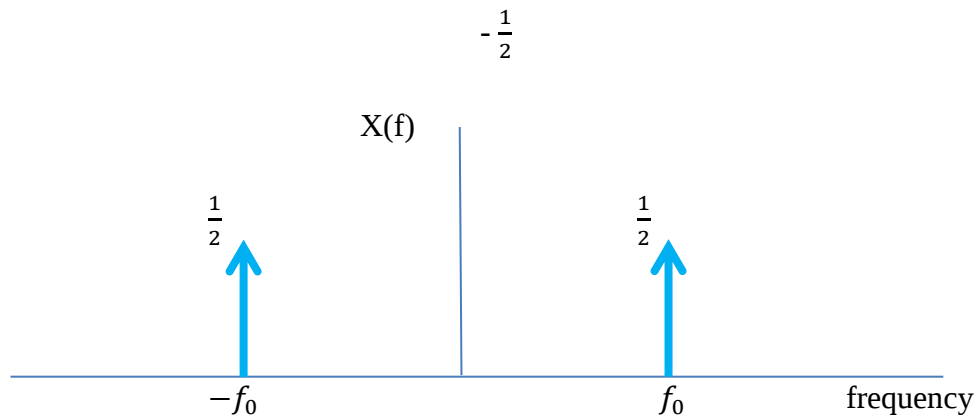


Figure 9.5

In Figure 9.5, sinusoidal signals [18] have only one spectral component at both positive and negative frequencies. This is to demonstrate a simple illustration of the time and frequency domain conversion. It is transmitting only one frequency, f_0 , as can be seen from the above figures, and this type of signal can be considered a tone. Most signals, however, must carry much more information than a single tone. A larger amount of information can be messaged across a band of frequencies called bandwidth. For example, the rectangular pulse carries information across its width and its length [7], as previously defined, can be now thought in terms of bandwidth or pulse width as pictured in Figure 9.6.

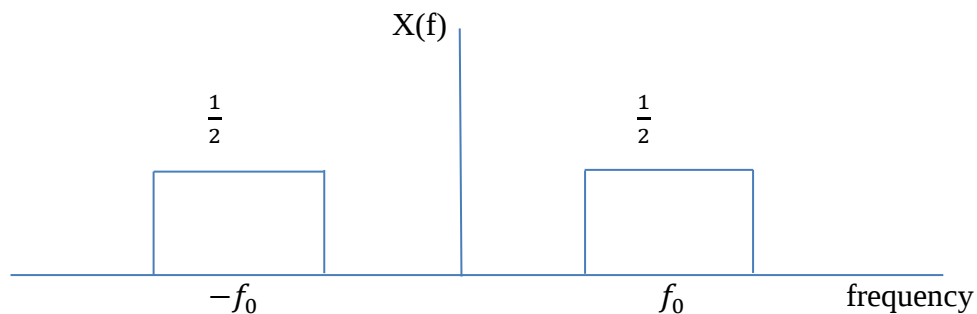


Figure 9.6

This is called a passband signal [7] and has a both positive and negative frequency bandwidth. A passband signal is creating by filtering the information through a process releasing only the required information onto a modulation carrier. The passband signal is created by the following [13].

$$x_c(t) = x(t)_m \cos(2\pi f_c t + \varphi(t))$$

The message portion [13] of the signal is $x(t)_m$ and is often called the envelope. The modulation carrier which up-converts the lower frequency message envelope to suitable higher frequency transmission is $\cos(2\pi f_c t + \varphi(t))$ where f_c is the center frequency and $\varphi(t)$ is the phase of the signal. The result is a modulated signal $x_c(t)$. The purpose of this short tutorial is to demonstrate that the cross-ambiguity function is developed from this form of signal, although in slightly different format as we shall demonstrate. The slightly different format alluded to is called the analytic signal. The analytic signal contains only positive frequency components unlike $x_c(t)$ which contains both positive and negative frequency components as previously illustrated in Figures 9.3, 9.4 and 9.5. And in developing the analytic signal, an important concept in signal processing must first be introduced and mathematically developed in order to fully understand the implications of using the analytic signal format. The following discussion therefore introduces the Hilbert Transform which suppresses the negative frequency components in a signal leaving only the positive side.

9.6 Hilbert Transform

The Hilbert Transform is a linear time-invariant function defined as [21]

$$H[g(t)] = g(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t - \tau)}{\tau} d\tau$$

Due the Transform being an improper integral, the Cauchy Principal Value (CPV) of the Hilbert Transform is re-expressed [21] as

$$H[g(t)] = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\frac{-1}{\varepsilon}}^{-\varepsilon} \frac{g(t - \tau)}{\tau} d\tau + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{g(t - \tau)}{\tau} d\tau \right)$$

First, we will conduct a Hilbert Transform on the cosine signal where

$$g(t) = \cos(2\pi f_c t + \varphi(t)) \text{ and } \tau = 2\pi f_c(\tau) + \varphi(\tau)$$

Plugging the cosine signal into the Hilbert Transform, we have

$$H[\cos(2\pi f_c t + \varphi(t))] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(t - \tau) + \varphi(t - \tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau$$

Using the following trigonometric relationship to expand the transform [37],

$$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) = \cos(y) \cos(x) + \sin(y) \sin(x)$$

and noticing that we can distribute and associate terms in the numerator as

$$\cos(2\pi f_c t - 2\pi f_c \tau + \varphi(t) - \varphi(\tau))$$

$$\cos((2\pi f_c t + \varphi(t)) - (2\pi f_c \tau + \varphi(\tau)))$$

we have [26],

$$\begin{aligned} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} \cos(2\pi f_c t + \varphi(t)) \\ &\quad + \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} \sin(2\pi f_c t + \varphi(t)) d\tau \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} \cos(2\pi f_c t + \varphi(t)) \\ &\quad + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} \sin(2\pi f_c t + \varphi(t)) d\tau \\ &= \frac{1}{\pi} \cos(2\pi f_c t + \varphi(t)) \int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} \\ &\quad + \frac{1}{\pi} \sin(2\pi f_c t + \varphi(t)) \int_{-\infty}^{\infty} \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau \end{aligned}$$

To make the integration easier for the sine part of the now-expanded transform, we make use of the relation [38],

$$\frac{1}{x} = \int_0^{\infty} e^{-xk} dk$$

This equals $\frac{1}{x}$ by the following. Given,

$$\int_0^{\infty} e^{-xk} dk$$

use substitution where,

$$u = -xk \text{ and } du = -xdk$$

$$\frac{k}{u} \int_0^{\infty} e^u du =$$

$$\frac{k}{u} e^u =$$

Re-substituting and evaluating, we have

$$\frac{-1}{x} e^{-xk} =$$

and integrating from 0 to infinity

$$\frac{1}{x}$$

We now continue to simplify the sine part of the Hilbert Transform since after expansion each sine and cosine part of the transform can be manipulated separately. Then we will simplify the cosine part. The above relation is now introduced into the sine part of the transform and since the sine function is an even function we can re-write the limits of integration at the same time [38]

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau = 2 \int_0^{\infty} \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau$$

$$= 2 \int_0^{\infty} \int_0^{\infty} e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau dk$$

Using integration by parts for the inner integral, where

$$\int u dv = uv - \int v du$$

and substituting [38]

$$\int_0^{\infty} e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau$$

the first step of the integration by parts process is assigning variables where $u = e^{-k(2\pi f_c(\tau) + \varphi(\tau))}$ and $dv = \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau$, and

$$du = -k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} d\tau$$

$$v = -\cos(2\pi f_c(\tau) + \varphi(\tau))$$

and substituting [38], we have

$$\begin{aligned} &= e^{-k(2\pi f_c(\tau) + \varphi(\tau))} [-\cos(2\pi f_c(\tau) + \varphi(\tau))] \\ &- \int [-\cos(2\pi f_c(\tau) + \varphi(\tau))] [-k e^{-k(2\pi f_c(\tau) + \varphi(\tau))}] d\tau \\ &= -e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) \\ &- \int k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \end{aligned}$$

The second step of integration by parts is assigning variables where $u = k e^{-k(2\pi f_c(\tau) + \varphi(\tau))}$ and $dv = \cos(2\pi f_c(\tau) + \varphi(\tau))$, and

$$du = -k^2 e^{-k(2\pi f_c(\tau) + \varphi(\tau))} d\tau$$

$$v = \sin(2\pi f_c(\tau) + \varphi(\tau))$$

and substituting, we have

$$\begin{aligned}
& \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
&= -e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) \\
&\quad - [k e^{-k(2\pi f_c(\tau) + \varphi(\tau))}] [\sin(2\pi f_c(\tau) + \varphi(\tau))] \\
&\quad - \int [\sin(2\pi f_c(\tau) + \varphi(\tau))] [-k^2 e^{-k(2\pi f_c(\tau) + \varphi(\tau))}] d\tau
\end{aligned}$$

Further, simplifying [37] and [38] we have,

$$\begin{aligned}
& \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
&= -e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) - k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\
&\quad - k^2 \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
& \quad \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
& \quad + k^2 \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
&= -e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) - k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\
& \quad (1 + k^2) \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
&= -e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) - k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\
& \quad \int e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\
&= \frac{-e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) - k e^{-k(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau))}{(1 + k^2)}
\end{aligned}$$

Evaluating the integral from 0 to infinity [38]

$$\frac{1}{1 + k^2}$$

Placing this result back into the original double integral, we have [37], [38] and [44] the following which can be resolved by integration tables

$$2 \int_0^{\infty} \frac{1}{1+k^2} dk = 2 \cdot \arctan(k) = 2 \cdot \frac{\pi}{2} = \pi$$

For the cosine part of the Hilbert Transform, we realize $\frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)}$ is an odd function where $f(-x) = -f(x)$ so the integral of this odd function $\int_{-\infty}^{\infty} f(x) = 0$. Even the phase components can be demonstrated to yield $\varphi(-x) = -\varphi(x)$ by the complex plane in Figure 9.2. Integration of this function is in Appendix 2.

Finally, the results will be substituted into the original equation which is re-stated here.

$$\begin{aligned} & H[\cos(2\pi f_c t + \varphi(t))] \\ &= \frac{1}{\pi} \cos(2\pi f_c t + \varphi(t)) \int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} + \frac{1}{\pi} \sin(2\pi f_c t \\ & \quad + \varphi(t)) \int_{-\infty}^{\infty} \frac{\sin(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau \end{aligned}$$

Substituting in the aforementioned results we have,

$$\begin{aligned} H[\cos(2\pi f_c t + \varphi(t))] &= \frac{1}{\pi} \cos(2\pi f_c t + \varphi(t)) \cdot 0 + \frac{1}{\pi} \sin(2\pi f_c t + \varphi(t)) \cdot \pi \\ &= \frac{1}{\pi} \sin(2\pi f_c t + \varphi(t)) \cdot \pi \\ &= \sin(2\pi f_c t + \varphi(t)) \end{aligned}$$

Thus the Hilbert Transform shifts the phase of cosine to sine.

9.7 The Analytic Signal

Since we have computed the Hilbert Transform we can now compute the analytic signal. The formula is in terms of the original signal and its Hilbert Transform [5]

$$\Psi(t) = g(t) + jH(g(t))$$

Substituting, we have

$$= \cos(2\pi f_c t + \varphi(t)) + j \sin(2\pi f_c t + \varphi(t))$$

Converting to complex exponential form

$$\begin{aligned} &= \frac{e^{j(2\pi f_c t + \varphi(t))} + e^{-j(2\pi f_c t + \varphi(t))}}{2} + j \frac{e^{j(2\pi f_c t + \varphi(t))} - e^{-j(2\pi f_c t + \varphi(t))}}{2j} \\ &= \frac{e^{j(2\pi f_c t + \varphi(t))} + e^{-j(2\pi f_c t + \varphi(t))}}{2} + \frac{e^{j(2\pi f_c t + \varphi(t))} - e^{-j(2\pi f_c t + \varphi(t))}}{2} \\ &= \frac{e^{j(2\pi f_c t + \varphi(t))} + e^{-j(2\pi f_c t + \varphi(t))} + e^{j(2\pi f_c t + \varphi(t))} - e^{-j(2\pi f_c t + \varphi(t))}}{2} \\ &= \frac{e^{j(2\pi f_c t + \varphi(t))} + e^{j(2\pi f_c t + \varphi(t))}}{2} \\ &= e^{j(2\pi f_c t + \varphi(t))} \end{aligned}$$

Thus, the analytic signal preserves only the positive frequency component of the signal and has a magnitude of one. No information is lost in this procedure, since the analytic signal maintains the carrier, amplitude and phase information of the signal, and is, therefore the preferred form in conducting computations.

At this point, we can introduce the rectangular pulse function, into the three major elements of its transformation: as a modulated signal, one phase-shifted by the Hilbert Transform, and a signal with its negative frequency component suppressed – the analytic signal ready for further computation. For our baseband message function $\Pi(t)$, we can determine its analytic signal function.

$$\Pi_c(t) = \Pi(t)\cos(2\pi f_c t + \varphi(t))$$

$$H(\Pi_c(t)) = \Pi(t)\sin(2\pi f_c t + \varphi(t))$$

$$\Psi_{\Pi(t)} = \Pi(t)\cos(2\pi f_c t + \varphi(t)) + j\Pi(t)\sin(2\pi f_c t + \varphi(t))$$

$$\Psi_{\Pi(t)} = \Pi(t)[\cos(2\pi f_c t + \varphi(t)) + j\sin(2\pi f_c t + \varphi(t))]$$

$$\Psi_{\Pi(t)} = \Pi(t)e^{j2\pi f_c t + \varphi(t)}$$

This analytic signal function is the function that we will use to compute the cross-ambiguity function in Section 12. The message and e^x components of the signal have been established whereby they can be further plugged-into the convolution integral. The analytic signal format not only preserves frequency and phase information, but also makes the convolution algebraically easier to resolve as we will see in Section 12.

10. From Convolution to Cross-Correlation

The convolution operator produced an output signal after an input signal was modulated by an impulse response in a linear time-invariant system. Convolution determines how much information the output receives from the input after the input has been weighted. Cross-correlation, on the other hand, is the processing of two *similar* signals [48] and determining how much they have in common. While convolution is the mathematical foundation of the cross-ambiguity function, the particular implementation of the cross-ambiguity function is to determine not just the amount of overlap, as is the case of convolution, but more specifically, the degree of association between these two *similar* signals with a time delay. This is similar to a matched filter where a defined signal template is compared to a proposed signal with similar characteristics. Thus, the “cross-“ in the cross-ambiguity function refers to the operation of cross-correlation. The convolution and cross-correlation operations solve slightly different problems and are therefore mathematically different. The major difference is that in cross-correlation there is no flipping of a signal, like the Faltung in convolution. However, we can derive the cross-correlation function from that of convolution, as the following derivation presents. We start [46] by re-introducing the convolution integral

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$$

Given the relation for the cross-correlation operation denoted by “ $\star$ ” and the “ $*$ ” in the superscript of the function $f^*(-t)$ denotes [46] the complex conjugate of that function, although the derivation works for non-complex operations as well [48]. The complex conjugate will be discussed in more detail in the next section.

$$(f \star g)(t) = f^*(-t) * g(t)$$

Substitute [46] into the convolution operation on the right hand side yielding

$$(f \star g)(t) = \int_{-\infty}^{\infty} f^*(-\tau)g(t - \tau)d\tau$$

Letting $\tau' = -\tau$ and $d\tau' = -d\tau$ [46] we have,

$$(f \star g)(t) = \int_{\infty}^{-\infty} f^*(\tau')g(t + \tau')(-d\tau')$$

$$(f \star g)(t) = \int_{-\infty}^{\infty} f^*(\tau')g(t + \tau')d\tau'$$

Let $\tau' = \tau$, and therefore $d\tau' = d\tau$ [46],

$$(f \star g)(t) = \int_{-\infty}^{\infty} f^*(\tau)g(t + \tau)d\tau$$

Thus, we now see the cross-correlation operation is an operation on two signals, whether they are real [48] or complex [46], separated by some time delay [46], and there is no longer any reversing of the signal.

In a later section when we process a pair of similar rectangular pulse signals we will find that since $\Pi(t)$ is an even function the convolution and cross-correlation functions have the same output [46] and this is given by the following relation [46]

$$(f \star g)(t) = (f * g)(t)$$

11. Complex Conjugate

The purpose of the complex conjugate in the cross-ambiguity function is ensure that ***if*** the envelope of a signal contains complex figures then there is a positive addition of phase

and amplitude information to the cross-correlation of the signal and avoid zero-averaging of the signal set. Taking the complex conjugate has the effect of reversing the direction of rotation of the complex exponential. The complex conjugate can be taken on either signal in the cross-ambiguity function since both signals are required to produce the positive contribution of phase and amplitude information. We demonstrate this using the following example. Signal One is a simple signal with constant amplitude, and Signal Two is a similar signal with constant amplitude. Each is represented by its exponential and trigonometric forms to illustrate the processing of the imaginary components.

$$e^{j(2\pi ft + \varphi(t))} = \cos(2\pi ft + \varphi(t)) + j \sin(2\pi ft + \varphi(t)) = s_1(t)$$

$$2e^{j(2\pi ft + \varphi(t))} = 2 \cos(2\pi ft + \varphi(t)) + 2j \sin(2\pi ft + \varphi(t)) = s_2(t)$$

The signal plot for $s_1(t)$ and $s_2(t)$ is as in Figure 11.1

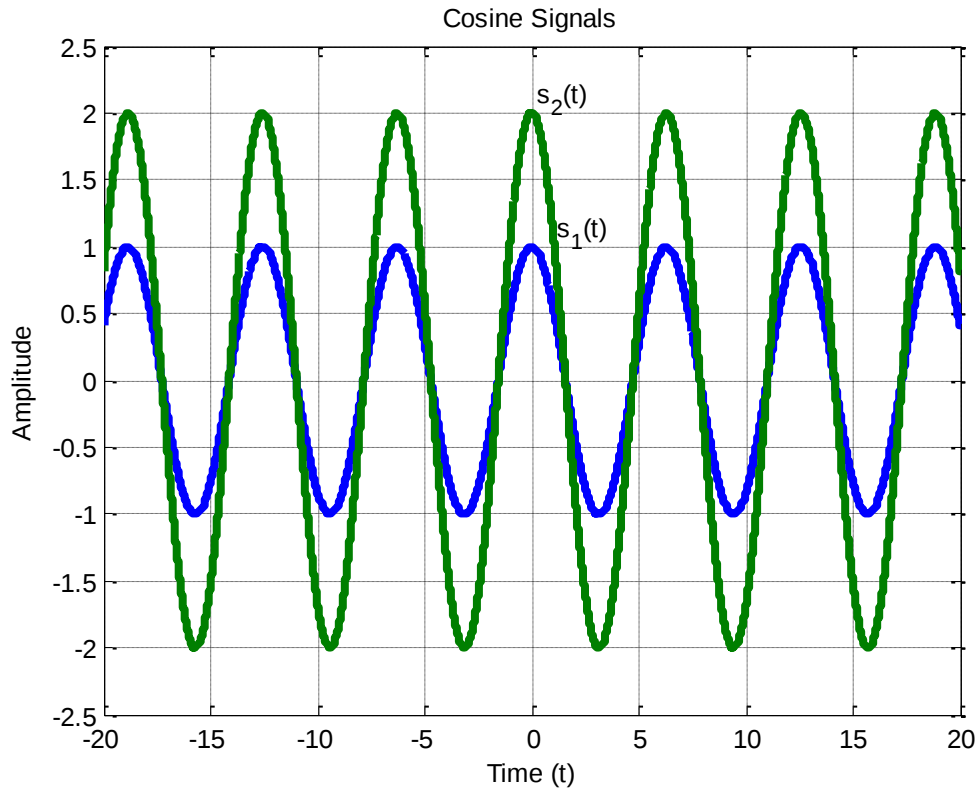


Figure 11.1

We now multiply the two signals and take the complex conjugate of $s_2(t)$ as $s_2^*(t)$ where the “*” is the complex conjugate notation for the signal. In other words

$$s_2^*(t) = 2e^{-j(2\pi ft + \varphi(t))} = 2\cos(2\pi ft + \varphi(t)) - 2j\sin(2\pi ft + \varphi(t))$$

Notice the imaginary components have been negated with a “-“. Taking the product of the signals $s_1(t)$ and $s_2^*(t)$

$$[(\cos(2\pi ft + \varphi(t)) + j\sin(2\pi ft + \varphi(t))][2\cos(2\pi ft + \varphi(t)) - 2j\sin(2\pi ft + \varphi(t))]$$

$$2(\cos(2\pi ft + \varphi(t)))^2 + 2j\sin(2\pi ft + \varphi(t))\cos(2\pi ft + \varphi(t)) - 2j\sin(2\pi ft + \varphi(t))\cos(2\pi ft + \varphi(t)) - 2(j\sin(2\pi ft + \varphi(t)))^2$$

$$= 2(\cos(2\pi ft + \varphi(t)))^2 - 2(j\sin(2\pi ft + \varphi(t)))^2$$

$$= 2[(\cos(2\pi ft + \varphi(t)))^2 + (\sin(2\pi ft + \varphi(t)))^2]$$

$$= 2$$

Thus, the complex conjugate operation removes the imaginary components of the product leaving only positive contributions in both phase and amplitude by both signals which is the magnitude (the length). In other words, the complex representations in the product produce a real number. The following Figures 11.2 and 11.3 demonstrate this positive contribution afforded by the complex conjugate. The product of the signals by the complex conjugate enables one to compute the magnitude of the signal and thus the power it contains.

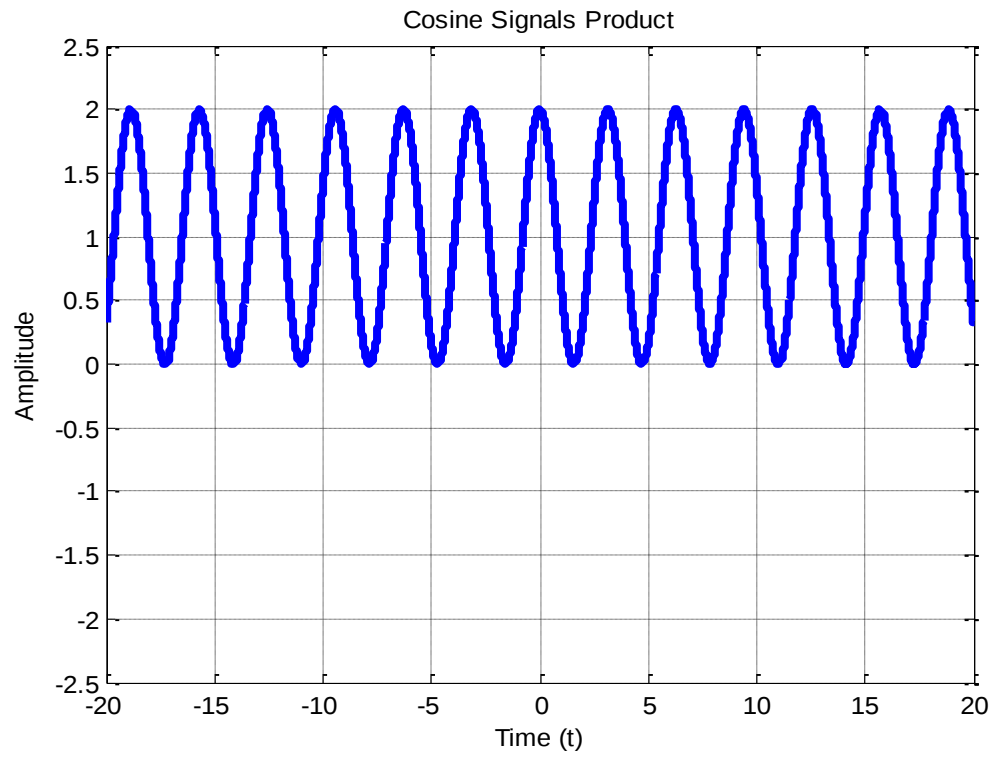


Figure 11.2

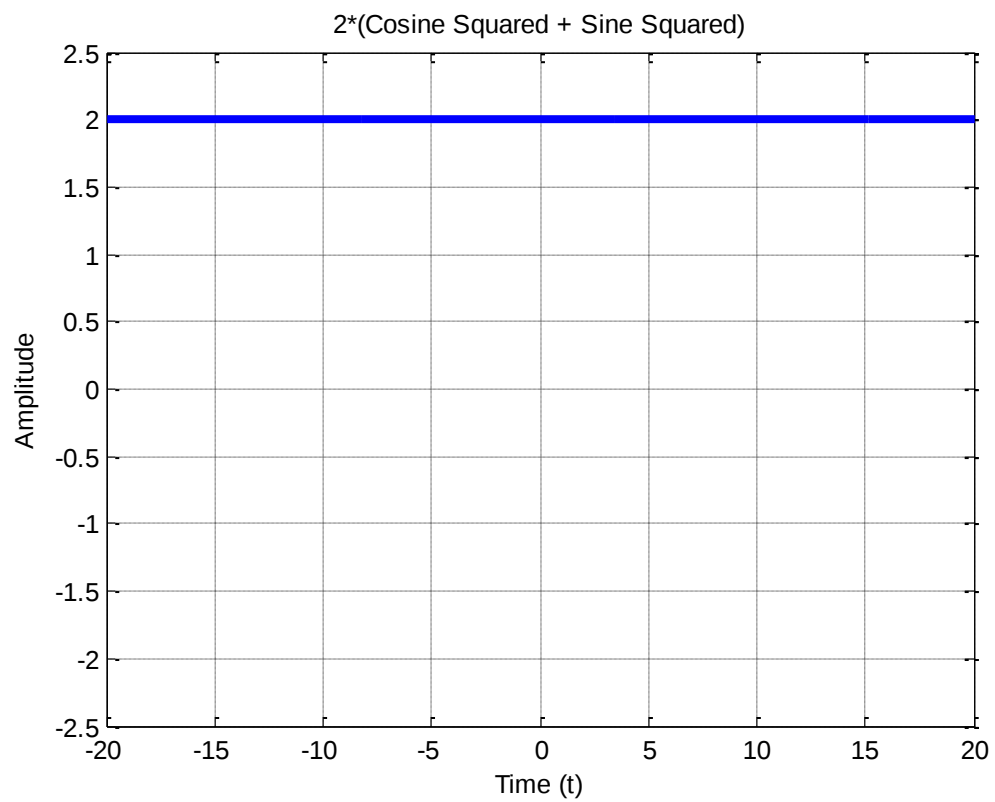


Figure 11.3

12. Compute the Cross-Ambiguity Function

The groundwork for the differing components underlying the cross-ambiguity function has been developed mathematically. This section implements this development to present the final form of the cross-ambiguity function. We begin by defining two different, but similar, signals in analytic signal format [49]. Signal $s_1(t)$ has a time delay of τ_d introduced into it. Signal $s_2(t)$ is attributed with the complex conjugate notation.

Let

$$f(t) = s_1(t - \tau_d)e^{j2\pi f_1 t} \quad \text{and}$$

$$g(t) = s_2^*(t)e^{j2\pi f_2 t}$$

The convolution integral is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x - u)du$$

Substitute both signals into the convolution integral yielding

$$(f * g)(x) = \int_{-\infty}^{\infty} s_1(u - \tau_d)e^{j2\pi f_1 u}s_2^*(x - u)e^{j2\pi f_2(x-u)}du$$

Derive cross-correlation as in Section 10,

$$= \int_{-\infty}^{\infty} s_1(-(u - \tau_d))e^{j2\pi f_1 u}s_2^*(x - u)e^{j2\pi f_2(x-u)}du$$

$$= \int_{-\infty}^{\infty} s_1(\tau_d - u)e^{j2\pi f_1 u}s_2^*(x - u)e^{j2\pi f_2(x-u)}du$$

$$t = \tau_d - u$$

$$dt = -du$$

$$= \int_{\infty}^{-\infty} s_1(t)e^{j2\pi f_1(\tau_d-t)}s_2^*(x - (\tau_d - t))e^{j2\pi f_2(x-(\tau_d-t))}(-dt)$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} s_1(t) e^{j2\pi f_1(\tau_d - t)} s_2^*(t + x - \tau_d) e^{j2\pi f_2(t + x - \tau_d)} dt \\
&\quad \tau = x - \tau_d \\
&= \int_{-\infty}^{\infty} s_1(t) e^{j2\pi f_1(\tau_d - t)} s_2^*(t + \tau) e^{j2\pi f_2(t + \tau)} dt \\
&= \int_{-\infty}^{\infty} s_1(t) e^{j2\pi f_1 \tau_d} e^{j2\pi f_1(-t)} s_2^*(t + \tau) e^{j2\pi f_2 t} e^{j2\pi f_2 \tau} dt \\
&= e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) e^{-j2\pi f_1 t} s_2^*(t + \tau) e^{j2\pi f_2 t} dt \\
&= e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) e^{-j2\pi f_1 t + j2\pi f_2 t} s_2^*(t + \tau) dt \\
&= e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) e^{j2\pi(f_2 - f_1)t} s_2^*(t + \tau) dt
\end{aligned}$$

Denote the apparent frequency difference between the two signals as ν

$$\begin{aligned}
\nu &= f_2 - f_1 \\
&= e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) e^{j2\pi \nu t} s_2^*(t + \tau) dt
\end{aligned}$$

Denoting this by $X(\tau, \nu)$, we have

$$X(\tau, \nu) = e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) s_2^*(t + \tau) e^{j2\pi \nu t} dt$$

To find the peak power, $|X(\tau, \nu)|$, that is indicative of the optimal solution between the time delay and frequency off-set, we take the absolute value of the equation by Parseval's Theorem.

$$|X(\tau, \nu)| = \left| e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \int_{-\infty}^{\infty} s_1(t) s_2^*(t + \tau) e^{j2\pi \nu t} dt \right|$$

$$|X(\tau, \nu)| = \left| e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \left| \int_{-\infty}^{\infty} s_1(t) s_2^*(t + \tau) e^{j2\pi \nu t} dt \right| \right|$$

Since

$$M = |c| = \sqrt{a^2 + b^2}$$

We can simplify the term taken outside the integral

$$\left| e^{j2\pi f_1 \tau_d} e^{j2\pi f_2 \tau} \right| = \left| e^{j2\pi f_1 \tau_d} \right| \left| e^{j2\pi f_2 \tau} \right|$$

$$e^{j2\pi f_1 \tau_d} = \cos(2\pi f_1 \tau_d) + j \sin(2\pi f_1 \tau_d)$$

$$a = \cos(2\pi f_1 \tau_d) \quad \text{and} \quad b = \sin(2\pi f_1 \tau_d)$$

$$M = \sqrt{\cos(2\pi f_1 \tau_d)^2 + \sin(2\pi f_1 \tau_d)^2} = 1$$

Likewise

$$M = \sqrt{\cos(2\pi f_2 \tau)^2 + \sin(2\pi f_2 \tau)^2} = 1$$

Finally,

$$|X(\tau, \nu)| = \left| \int_{-\infty}^{\infty} s_1(t) s_2^*(t + \tau) e^{j2\pi \nu t} dt \right|$$

As indicated, $|X(\tau, \nu)|$, is the peak (highest degree of similarity) of the cross-ambiguity function, as reference [47] states, “for a *combined* time and frequency shift”. This new function permits computation of magnitudes at various time and frequency measurements. The largest peak indicating that a particular pair of time-shifts and frequency off-sets produces a largest magnitude of the set of all pairs. This allows the analyst to further infer important derived calculations like emitter location.

13. Cross-Ambiguity Function Assessment of the Rectangular Pulse

In Section 12, we completed the development of the cross-ambiguity function. The purpose of this section is to implement a specific signal set into the CAF and simplify it to its final form. For this specific application, the two similar signals under investigation will be normalized rectangular pulses. They are the baseband message envelopes and can be substituted directly into the CAF. As noted before, the rectangular pulse function is defined as

$$\Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

For the two normalized signals we define them as [43]

$$s_1(t) = \frac{1}{\sqrt{T}} \Pi\left(\frac{t}{T}\right)$$

$$s_2^*(t + \tau) = \frac{1}{\sqrt{T}} \Pi^*\left(\frac{t + \tau}{T}\right)$$

Substituting into the CAF, we have

$$|X(\tau, \nu)| = \left| \frac{1}{T} \int_{-\infty}^{\infty} \Pi\left(\frac{t}{T}\right) \Pi^*\left(\frac{t + \tau}{T}\right) e^{j2\pi\nu t} dt \right|$$

We now perform the integration in a similar manner as in Sections 8.4 and 8.5. The limits of integration are from $\frac{-T}{2}$ to $\frac{T}{2} - \tau$ or $\tau \geq 0$ as in Figure 13.1.

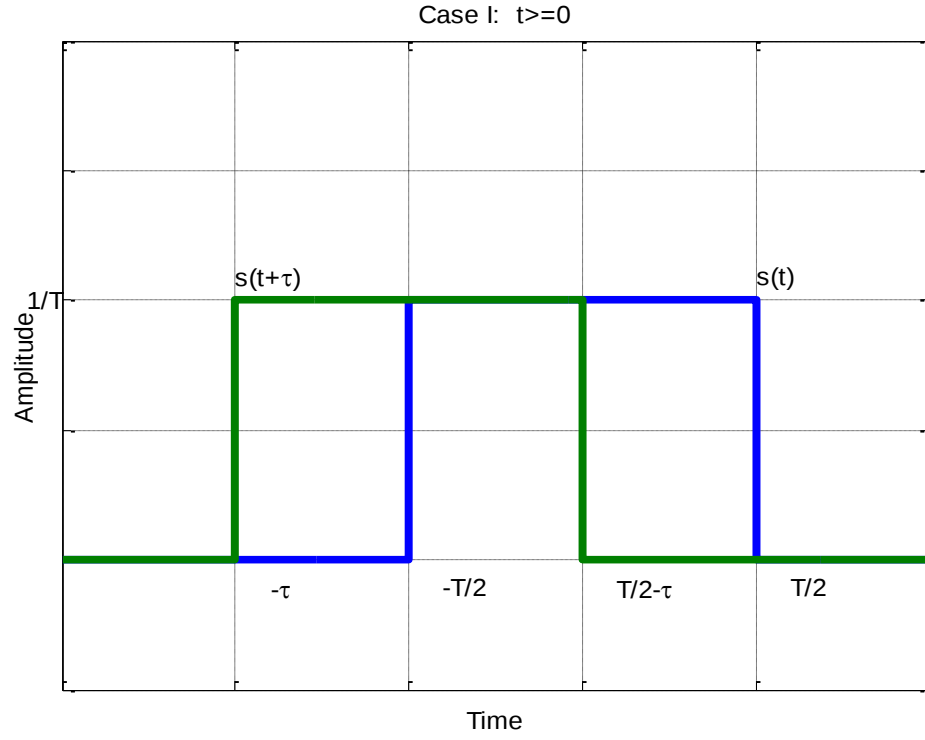


Figure 13.1

$$= \left| \int_{-\frac{T}{2}}^{\frac{T}{2}-\tau} e^{j2\pi vt} dt \right|$$

$$= \left| \frac{1}{T} \frac{1}{j2\pi v} [e^{j2\pi v(\frac{T}{2}-\tau)} - e^{j2\pi v(\frac{-T}{2})}] \right|$$

$$= \left| \frac{1}{T} \frac{1}{j2\pi v} [e^{j\pi v(T-2\tau)} - e^{-j\pi vT}] \right|$$

$$= \left| \frac{1}{T} \frac{1}{j2\pi v} [e^{j\pi vT-2j\pi v\tau} - e^{-j\pi vT}] \right|$$

$$= \left| \frac{1}{T} \frac{1}{j2\pi v} [e^{j\pi vT} e^{-2j\pi v\tau} - e^{-j\pi vT}] \right|$$

$$= \left| \frac{1}{T} \frac{1}{j2\pi v} [e^{j\pi vT} e^{-j\pi v\tau-j\pi v\tau} - e^{-j\pi vT}] \right|$$

$$\begin{aligned}
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} \frac{e^{-j\pi\nu\tau}}{e^{j\pi\nu\tau}} - e^{-j\pi\nu T}] \right| \\
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} \frac{e^{-j\pi\nu\tau}}{e^{j\pi\nu\tau}} - e^{-j\pi\nu T} \frac{e^{j\pi\nu\tau}}{e^{j\pi\nu\tau}}] \right| \\
&= \left| \frac{1}{e^{j\pi\nu\tau}} \right| \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} e^{-j\pi\nu\tau} - e^{-j\pi\nu T} e^{j\pi\nu\tau}] \right| \\
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu(T-\tau)} - e^{-j\pi\nu(T-\tau)}] \right|
\end{aligned}$$

In the case $\tau \geq 0$ we can write $|\tau| = \tau$, to obtain

$$\begin{aligned}
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu(T-|\tau|)} - e^{-j\pi\nu(T-|\tau|)}] \right| \\
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} (T - |\tau|) \frac{2j\sin(\pi\nu(T-|\tau|))}{2j\pi\nu(T-|\tau|)} \right|
\end{aligned}$$

The magnitude of the complex exponential $|e^{-j\pi\nu\tau}|$ is one as was previously developed in Section 12, thus

$$|X(\tau, \nu)| = \left| \frac{1}{T} (T - |\tau|) \frac{\sin(\pi\nu(T-|\tau|))}{\pi\nu(T-|\tau|)} \right|$$

Given the relation where $\text{sinc}(x)$ [44], the sine cardinal

$$\frac{\sin(\pi x)}{\pi x} = \text{sinc}(x)$$

$$|X(\tau, \nu)| = \left| \frac{(T-|\tau|)}{T} \text{sinc}(\nu(T-|\tau|)) \right|$$

For the limits of integration from $\frac{T}{2}$ to $\frac{-T}{2} - \tau$ or $\tau < 0$ as in Figure 13.2.

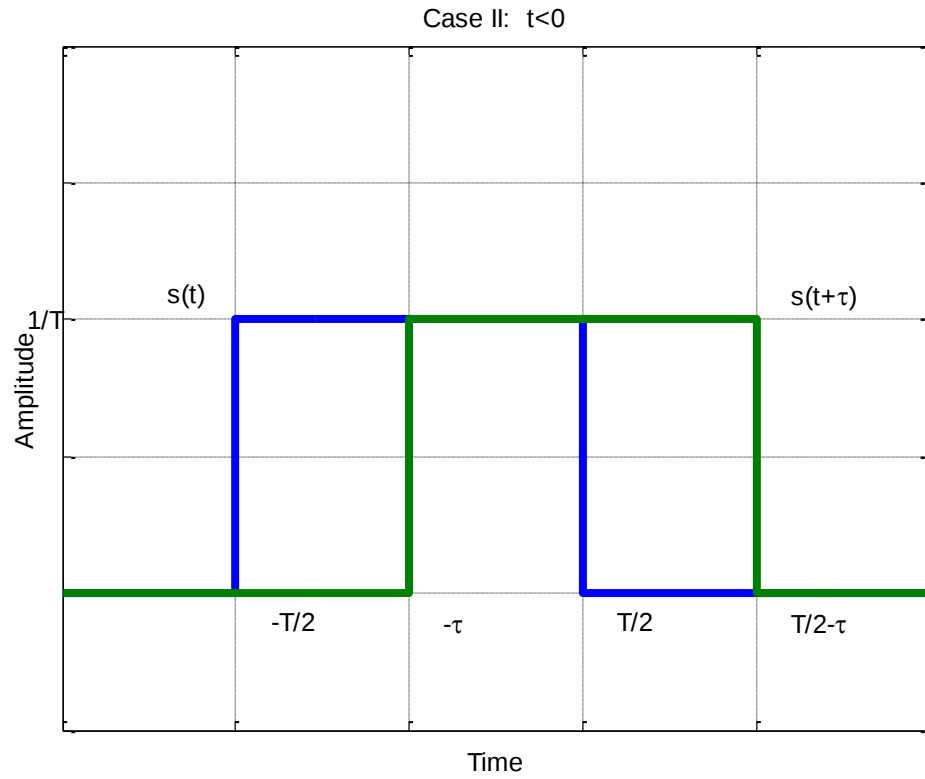


Figure 13.2

$$\begin{aligned}
 |X(\tau, \nu)| &= \left| \frac{1}{T} \int_{\frac{-T}{2}-\tau}^{\frac{T}{2}} e^{j2\pi\nu t} dt \right| \\
 &= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j2\pi\nu(\frac{T}{2})} - e^{j2\pi\nu(\frac{-T}{2}-\tau)}] \right| \\
 &= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} - e^{j\pi\nu(-T-2\tau)}] \right| \\
 &= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} - e^{-j\pi\nu T - 2j\pi\nu\tau}] \right| \\
 &= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} - e^{-j\pi\nu T} e^{-2j\pi\nu\tau}] \right|
 \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} - e^{-j\pi\nu T} e^{-j\pi\nu\tau - j\pi\nu\tau}] \right| \\
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} - e^{-j\pi\nu T} \frac{e^{-j\pi\nu\tau}}{e^{j\pi\nu\tau}}] \right| \\
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} \frac{e^{j\pi\nu\tau}}{e^{j\pi\nu\tau}} - e^{-j\pi\nu T} \frac{e^{-j\pi\nu\tau}}{e^{j\pi\nu\tau}}] \right| \\
&= \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu T} e^{j\pi\nu\tau} - e^{-j\pi\nu T} e^{-j\pi\nu\tau}] \right| \\
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu(T+\tau)} - e^{-j\pi\nu(T+\tau)}] \right|
\end{aligned}$$

In the case $\tau < 0$ we can write [6] $|\tau| = -\tau$ or $-|\tau| = \tau$

$$\begin{aligned}
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} \frac{1}{j2\pi\nu} [e^{j\pi\nu(T-|\tau|)} - e^{-j\pi\nu(T-|\tau|)}] \right| \\
&= |e^{-j\pi\nu\tau}| \left| \frac{1}{T} (T - |\tau|) \frac{2j\sin(\pi\nu(T-|\tau|))}{2j\pi\nu(T-|\tau|)} \right|
\end{aligned}$$

Again, the magnitude of the complex exponential $|e^{-j\pi\nu\tau}|$ is one, imply that

$$|X(\tau, \nu)| = \left| \frac{1}{T} (T - |\tau|) \frac{\sin(\pi\nu(T-|\tau|))}{\pi\nu(T-|\tau|)} \right|$$

Given the relation that $\text{sinc}(x)$ [44] is the normalized sine cardinal

$$\frac{\sin(\pi x)}{\pi x} = \text{sinc}(x)$$

This simplifies the subject expression, [43] as

$$|X(\tau, \nu)| = \left| \frac{(T-|\tau|)}{T} \text{sinc}(\nu(T-|\tau|)) \right|$$

which is the same expression as for the first limits of integration.

Thus, for both limits of integration, we characterize the CAF as

$$|X(\tau, \nu)| = \left| \frac{(T-|\tau|)}{T} \text{sinc}(\nu(T-|\tau|)) \right|$$

There are two main features of the result. One is the $\left| \frac{(T-|\tau|)}{T} \right|$ factor that produces the time domain aspect of the CAF. The second feature is the $|\text{sinc}(\nu(T-|\tau|))|$ that produces the frequency domain dimension of the CAF. Each feature can be demonstrated by setting τ and ν equal to zero in the CAF. By setting the $\tau = 0$ we can produce what is called the Doppler Cut giving us a view of the frequency domain produced by the sinc function. Conversely, by setting $\nu = 0$ we can produce what is called the Delay Cut. Both enable us to examine the process of the CAF in a clear two-dimensional fashion.

Using MATLAB, Figure 13.3 is the Doppler Cut view afforded by setting $\tau = 0$. This means there is no time delay between two collectors receiving the purported similar signal. This affords a straightforward analysis of the frequency off-set. First we set $\tau = 0$ in the CAF

$$|X(0, \nu)| = \left| \frac{(T-|0|)}{T} \text{sinc}(\nu(T-|\tau|)) \right| = |\text{sinc}(\nu(T))|$$

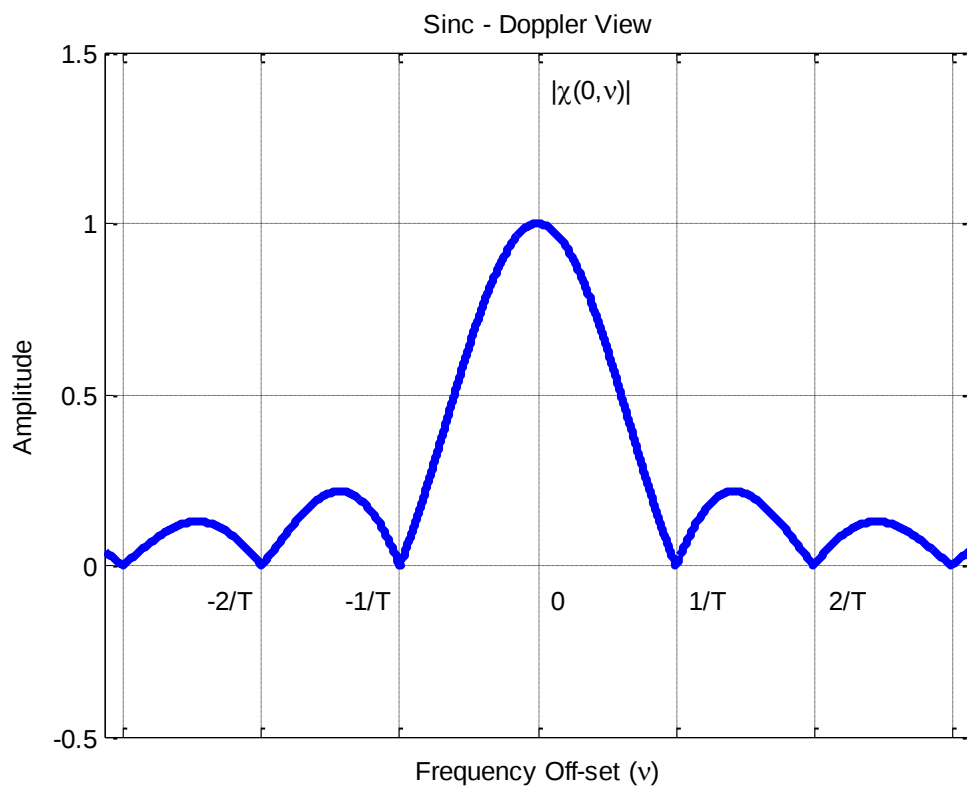


Figure 13.3

Notice that we have a simple view of the frequency off-set that tells us several things. The peak power occurs at a Doppler Shift of zero. Off-peak values of amplitude occur at Doppler Shifts containing some values. This means there is relative motion between three different entities: the two receivers and the transmitting source. Power values are zero at the null crossings for integer multiples of the pulse's bandwidth.

Again using MATLAB, Figure 13.4 is the Delay Cut view afforded by setting $\nu = 0$. This means there is no frequency off-set between two collectors receiving the purported similar signal. This affords a straightforward analysis of the time delay. In this case, we set $\nu = 0$ in the CAF

$$|X(\tau, 0)| = \left| \frac{(T-|\tau|)}{T} \text{sinc}(0(T-|\tau|)) \right| = \left| \frac{(T-|\tau|)}{T} \right|$$

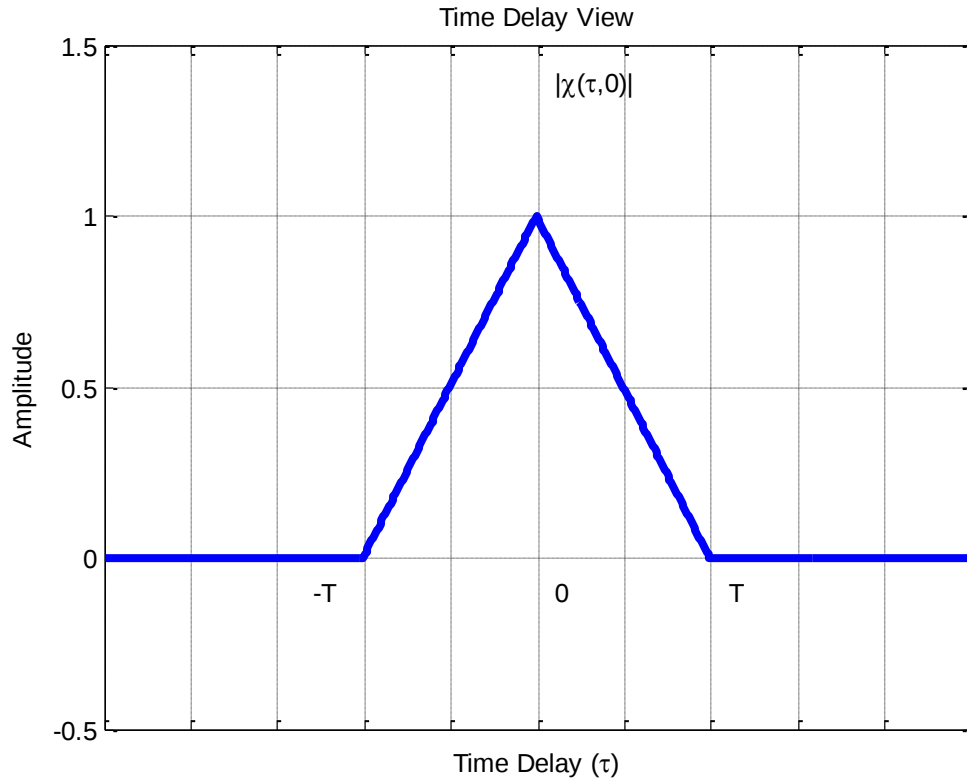


Figure 13.4

This figure should be a familiar form. Given the cross-correlation of the two normalized rectangular pulses, the output yields a triangular function at a normalized peak power of one.

14. A Practical Application

In this section, we will present a practical application of the cross-ambiguity function. We will introduce the parameters of a naval radar into the CAF and display it. Also we will generate a “Doppler Cut” and a “Delay Cut” of the CAF to more closely examine its features. The purpose of this is two-fold: it shows that real world numbers act according to our mathematical predictions as previously developed, and graphically shows the results of a zero time delay and a zero frequency off-set offering a practical view of the models presented in Section 13. The second demonstration will be a sample calculation of a non-zero time delay and frequency off-set to illustrate that the combination of parameters yield power results that fall-off the peak of the zero-based figures.

The operational scenario constructed here is a geometric search and rescue model of two low-altitude airborne collectors – unmanned air vehicles and a target carrying an actively transmitting AN/SPS-52. The two collector architecture offers the opportunity to compare the reception of a likely similar radar pulse – the cross-correlation notion. Thus, one collector will see the radar pulse as $s(t)$ and the other collector will see it as $s(t + \tau)$. Recall, this is the signal processing definition used to develop the idea of cross-correlation in Section 12. Additionally, our operational scenario will have one collector moving with some relative velocity to the other collector thus providing the ability to measure the frequency of the received pulse at slightly different perceived frequencies. In others words, the framework allows us to develop a frequency off-set or Doppler figure, ν , but the final Doppler figure will always be assumed in terms of the radial velocity referenced to the target transmitter. In this scenario, the target will be assumed to be (relatively) stationary.

The SPS-52 is a long range air search radar fitted on many (now older) US naval and Allied platforms [15]. Those radars with “long range” missions often had relatively large pulse widths (PW). In this long range operation, the AN/SPS-52 transmitted with a 10 microsecond pulse width and a (variable) 1000 pulses per second (PPS) pulse repetition frequency (PRF) or sometimes called the pulse repetition rate (PRR)[15]. The operating frequency, f_{ST} will be 3000 MHz [15]. It will also be assumed that the two collectors are flying in such a position that the distance between the first collector and the target, and the distance between the second collector and the target will be different. This allows the generation of the timed delay based on this difference. Additionally, the collectors also have the ability to measure the pulse width, PRF and perceived frequency of the radar’s transmitted pulse. Each collector and its geometry, induces slight measurement differences and errors. This explains the idea of using cross-correlation for seemingly similar signals. The operational geometry is shown in the following figure, Figure 14.1.

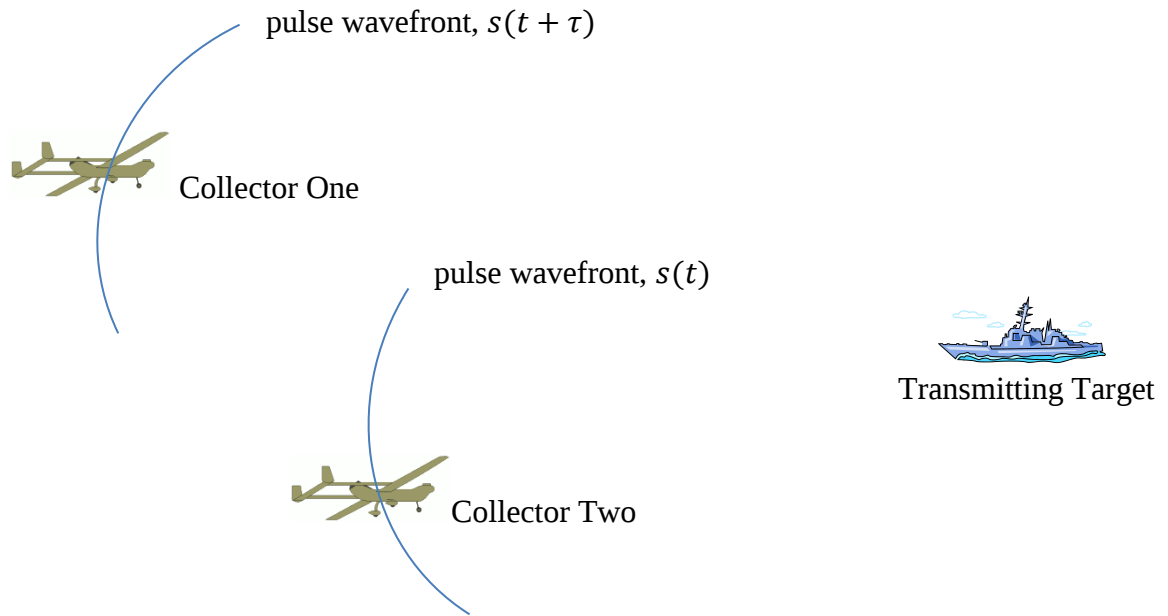


Figure 14.1

As seen in Figure 14.1 a single pulse wavefront impinges upon the two collectors at different times based on their relative distance. For this problem, we will calculate the time delay, τ , according to a straight left-to-right reference. Thus, the distance between the two collectors will be 2000 meters, d . The speed of light figure used in this calculation is 299,792,458 meters per second, r . Using the relation

$$\tau = \frac{d}{r}$$

we have,

$$\begin{aligned} \tau &= \frac{2000 \text{ meters}}{299,792,458 \text{ meters per second}} \\ &= 0.000006671 \text{ seconds} \end{aligned}$$

For the frequency off-set we will use the relation in Section 9.1 to calculate,

$$\nu = \Delta f = f_{ST} \frac{v_{MR}}{c}$$

where $c = 299,792,458$ meters per second, $f_{ST} = 3000$ MHz and $v_{MR} = 13.4$ meters per second. The v_{MR} has one collector moving at 13.4 meters per second faster than the second. Also, the negative sign in front of the Doppler relation is now positive since we are assuming the collectors are moving towards the target. Substituting, we have,

$$\begin{aligned} \nu &= 3,000,000,000 \text{ Hz} \frac{13.4 \text{ meters per second}}{299,792,458 \text{ meters per second}} \\ &= 134.0927 \text{ Hz} \end{aligned}$$

14.1 Results

Part one of the results is to present the ideal CAF. Using MATLAB's powerful `ambgfun` in the Phased Array System ToolBox, it is presented below as Figure 14.2.

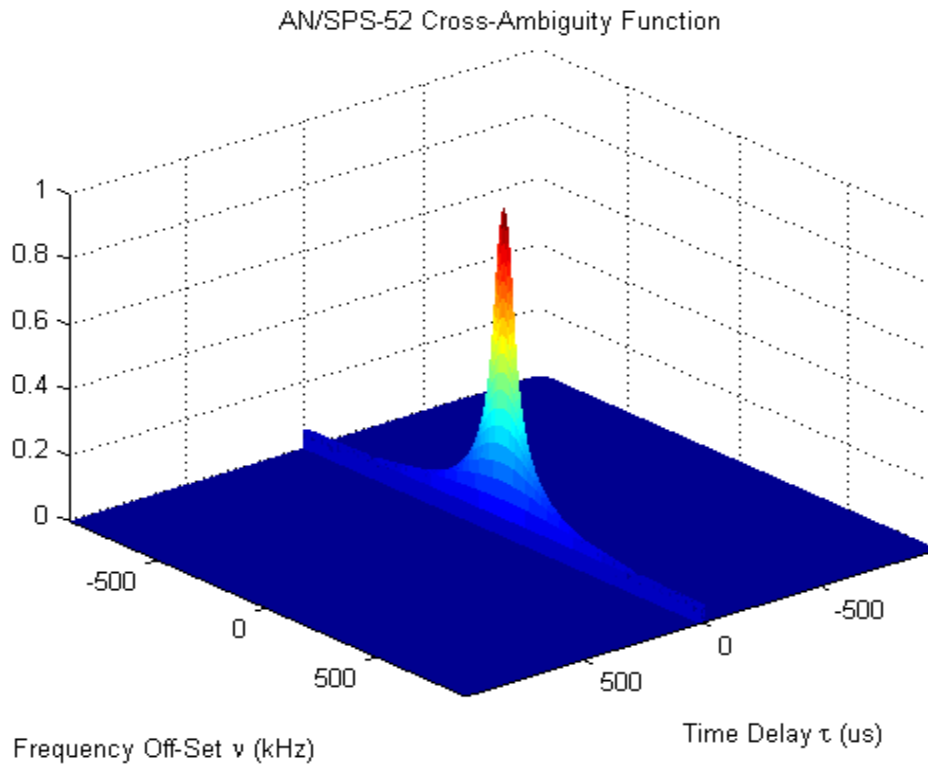


Figure 14.2

We see a distinctive peak at a time delay and Doppler shift of zero. Falling off rapidly, representing infinite combinations of other time delays and Doppler shifts each corresponding to their own power values.

Part one of the results also presents the Delay and Doppler Cut for the subject radar. The Doppler Cut is presented first. This assumes the time delay is zero. Figure 14.3 below refers.

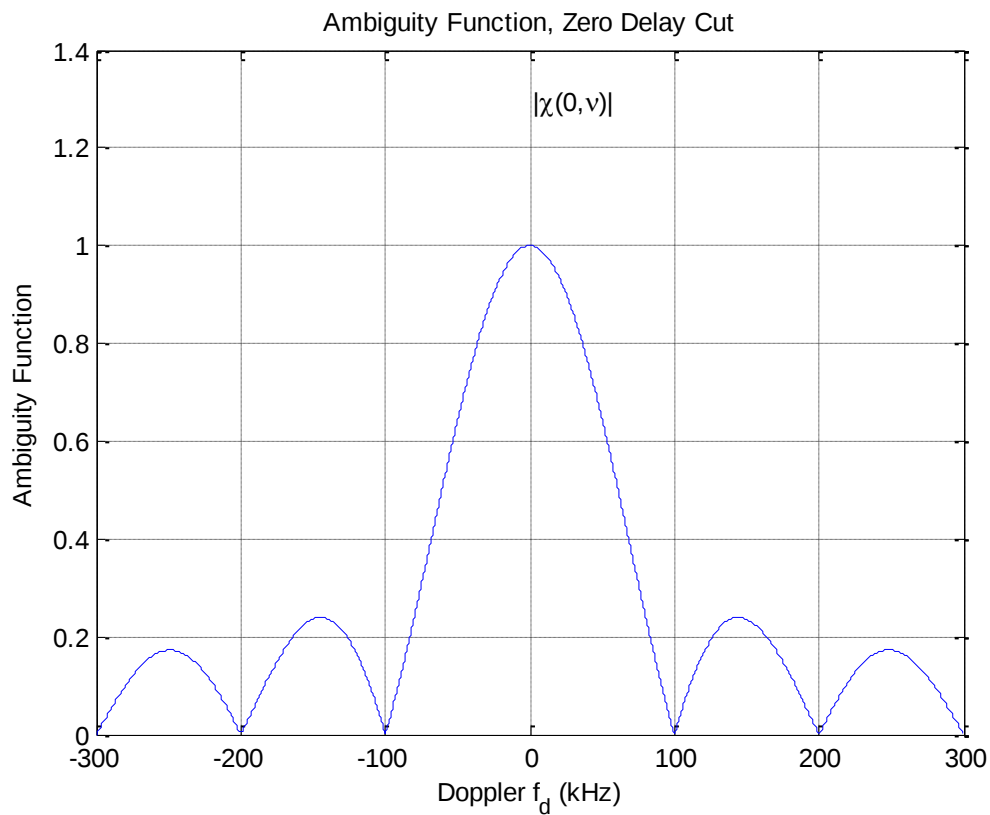


Figure 14.3

Like the analysis performed in Section 13, Figure 14.3 affords a look at the sinc function of the Doppler Shift. The amplitude is one and each frequency null corresponds to the positive and negative of the bandwidth at integer intervals for every sidelobe. As such, each lobe contains less and less power as predicted by the sinc function.

The second view is the Delay Cut where it is assumed the Doppler Shift is zero. This graphic is presented as Figure 14.4 below.

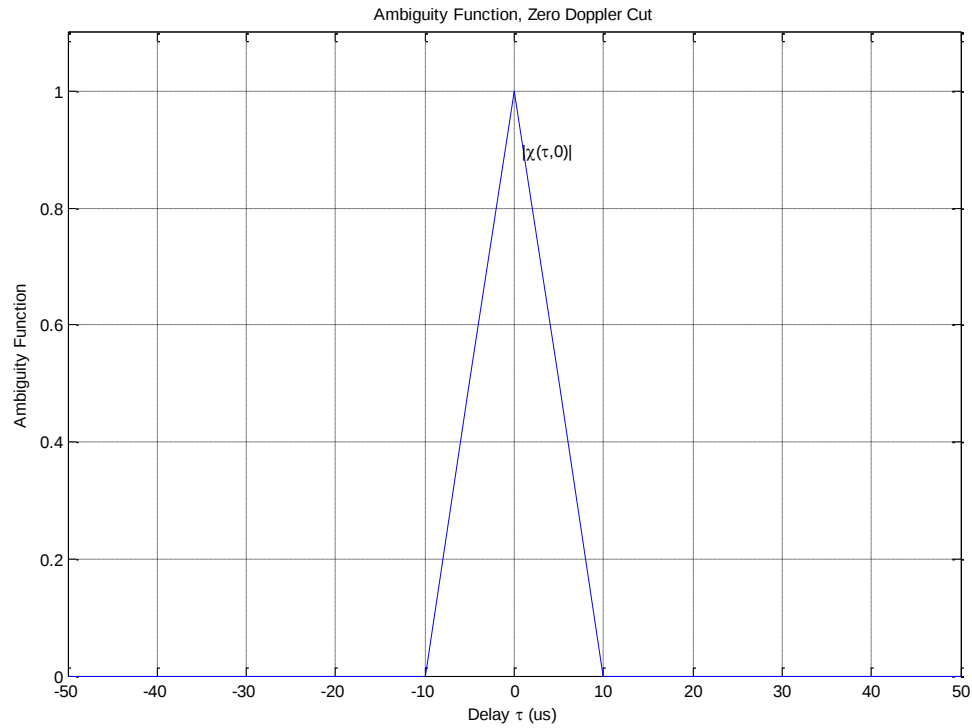


Figure 14.4

This graphic was processed at the appropriate Nyquist rate for the given pulse width. We can see, like in Section 13, the delay is accounted for between the 10 microsecond bandwidth boundaries, both positive and negative. The amplitude of the Doppler Cut is one.

The next validation of our model comes with substituting in the time delay and frequency offset into the CAF, and the evaluation.

$$\begin{aligned}
 & |X(0.000006671, 134.0927)| \\
 &= \left| \frac{(0.00001 - |0.000006671|)}{0.00001} \text{sinc}(134.0927(0.00001 - |0.000006671|)) \right| \\
 &= |0.3329 \text{sinc}(0.000446395)| \\
 &= 0.3329
 \end{aligned}$$

This represents the power, in this combination, of time delay and frequency offset. The point on the surface of the CAF representing the time delay, frequency off-set and power is illustrated in Figure 14.5 below.

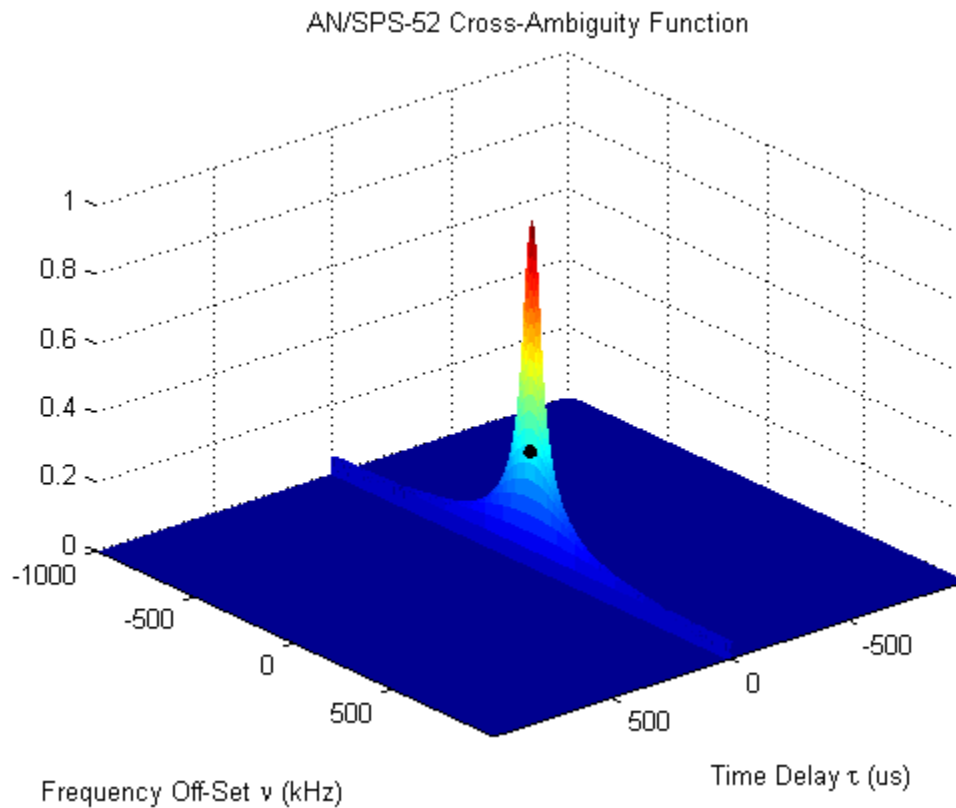


Figure 14.5

14.2 Operational View of the CAF Features

The CAF has been exhaustively developed with precision and detail. However, stepping back and seeing how these features of time delay, frequency offset and power fit take on physical meaning, in the overall operational picture, is especially instructive and brings to visual closure the work undertaken in this thesis. We start by revisiting Figure 14.1 and modify it to our purposes here as Figure 14.6 to illustrate at a higher level, the mathematical-architectural concepts that bridge the gap to geolocation. The time delay and frequency off-set are not just a point, but an infinite amount of points. The time delay, for example, is a number that represents the time difference that the pulse was received at the two collectors. This time difference is referred to as the time difference of arrival (TDOA) and can be represented as a line that represents points on the ground that could be where the target transmitter is. Likewise, the same view applies to the

frequency off-set or frequency difference of arrival (FDOA). The large symmetric arcs in Figure 14.6 represent all the possible locations on the ground where the target transmitter could be transmitting [19]. Where the two difference of arrival lines intersect is the probable location of the target transmitter as demonstrated by Figure 14.6.

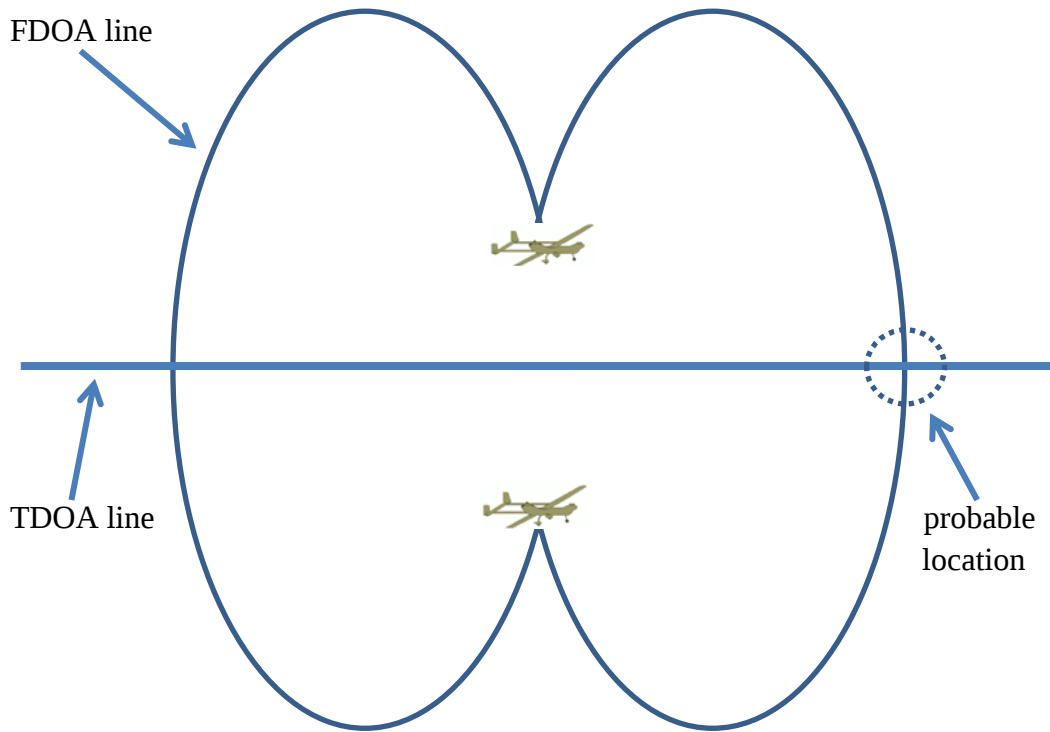


Figure 14.6

It is obvious to the reader that there are two intersecting points, but advanced geolocation techniques are able to discern the probable location between the two and produce a confidence ellipse around the probable true location. The intersection notated as the “probable location” would be represented in the power spectrum as the CAF peak as we have developed it previously in Section 12, page 67.

15. Final Comments and Recommendations

This concludes my study of the mathematical definition, development and validation of the cross-ambiguity function. Mathematical concepts such as convolution operator, the Euler Identity, complex conjugate, Hilbert Transform, sinc function, advanced trigonometric and integration techniques. Most of these concepts have been exhaustive

derivations with the goal to not only add to library of knowledge of the cross-ambiguity function, but offer a deeper understanding of the mathematical precepts that, if elucidated properly, can translate more easily in interpreting advanced signal processing processes.

This thesis addressed the one-pulse model. This work should be extended to, in both the discrete and continuous cases, a series of pulses. While many of the techniques would be similar, additional mathematical concepts would be explored to explain them in better detail as we have done here in the single pulse case.

Appendix 1

Truncated and Rectangular Pulse Convolution MATLAB Code

```
t = -4:0.01:5;      % time domain

dt = 0.01;          % delta t

x = rectpuls(t);     % MATLAB square pulse

h = truncatede(t+4); % truncated pulse

y2 = conv(x,h) * dt; % convolution operation

y2 = y2(1:length(t)); % length of output signal

plot(t,y2,'LineWidth',3),grid
xlabel('Time')
ylabel('Amplitude'),...
title('Output Signal y(t)')
axis ([-3 4 -0.5 2])
```

Rectangular Pulses Convolution MATLAB Code

```
t = -1.5:0.01:2.5;  % time domain

dt = 0.01;          % delta t

x = rectpuls(t);     % MATLAB square pulse

h = rectpuls(t+1);   % MATLAB square pulse shifted

y2 = conv(x,h) * dt; % convolution operation

y2 = y2(1:length(t)); % length of output signal

plot(t,y2,'LineWidth',3),grid
xlabel('Time')
ylabel('Amplitude'),...
title('Output Signal y(t)')
axis ([-3 4 -0.5 2])
```

Appendix 2

Hilbert Transform of Cosine Function

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau = 2 \int_0^{\infty} \frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)} d\tau$$

$$= 2 \int_0^{\infty} \int_0^{\infty} e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau ds$$

Since this is an improper integral the Cauchy Proper Value must be used

$$\int_0^{\infty} e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau$$

where $u = e^{-s(2\pi f_c(\tau) + \varphi(\tau))}$ and $dv = \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau$

First step of integration by parts process is

$$du = -s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} d\tau$$

$$v = \sin(2\pi f_c(\tau) + \varphi(\tau))$$

$$\begin{aligned} &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\ &\quad - \int \sin(2\pi f_c(\tau) + \varphi(\tau)) [-s e^{-s(2\pi f_c(\tau) + \varphi(\tau))}] d\tau \\ &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\ &\quad + \int s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau \end{aligned}$$

The second step of integration by parts is

Where $u = s e^{-s(2\pi f_c(\tau) + \varphi(\tau))}$ and $dv = \sin(2\pi f_c(\tau) + \varphi(\tau)) d\tau$

$$du = -s^2 e^{-s(2\pi f_c(\tau) + \varphi(\tau))} d\tau$$

$$v = -\cos(2\pi f_c(\tau) + \varphi(\tau))$$

$$\begin{aligned} & \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\ &+ [s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} [-\cos(2\pi f_c(\tau) + \varphi(\tau))]] \\ &- \int -s^2 e^{-s(2\pi f_c(\tau) + \varphi(\tau))} [-\cos(2\pi f_c(\tau) + \varphi(\tau))] d\tau \end{aligned}$$

Simplifying we have,

$$\begin{aligned} & \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) - s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) \\ &- s^2 \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ & \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ &+ s^2 \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\ &- s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) \\ &(1 + s^2) \int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau \\ &= e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) \\ &- s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) \end{aligned}$$

$$\int e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau)) d\tau$$

$$= \frac{e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \sin(2\pi f_c(\tau) + \varphi(\tau)) - s e^{-s(2\pi f_c(\tau) + \varphi(\tau))} \cos(2\pi f_c(\tau) + \varphi(\tau))}{(1 + s^2)}$$

Evaluating the integral from 0 to infinity, we have

$$\frac{s}{1 + s^2}$$

Placing this result into the double integral, we have the following which can be resolved by integration table

$$\int_{-\infty}^{\infty} \frac{s}{1 + s^2} ds$$

By substitution,

$$u = 1 + s^2 \text{ and } du = 2s ds$$

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{u} du$$

$$\frac{1}{2} \ln|u|$$

back substituting

$$\frac{1}{2} \ln|1 + s^2|$$

Integrating from $-\infty$ to ∞ , we find the result is undefined, and this is expected since $\frac{\cos(2\pi f_c(\tau) + \varphi(\tau))}{2\pi f_c(\tau) + \varphi(\tau)}$ is an odd function.

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